JOURNAL OF FORMALIZED MATHEMATICS
Volume 9, Released 1997, Published 2003
Inst. of Computer Science, Univ. of Białystok

Algebraic Operation on Subsets of Many Sorted Sets

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MML Identifier: CLOSURE3.

WWW: http://mizar.org/JFM/Vol9/closure3.html

The articles [13], [5], [16], [12], [17], [2], [4], [3], [7], [6], [14], [15], [1], [10], [8], [9], and [11] provide the notation and terminology for this paper.

1. Preliminaries

Let S be a non empty 1-sorted structure. One can verify that the 1-sorted structure of S is non empty.

The following three propositions are true:

(1) For every non empty set I and for all many sorted sets M, N indexed by I holds $M+\cdot N = N$.

(2) Let I be a set, M, N be many sorted sets indexed by I, and F be a family of many sorted subsets indexed by M. If $N \in F$, then $\bigcap|{:}F{:}| \subseteq N$.

(3) Let S be a non void non empty many sorted signature, M_1 be a strict non-empty algebra over S, and F be a family of many sorted subsets indexed by the sorts of M_1. Suppose $F \subseteq \mathrm{SubSorts}(M_1)$. Let B be a subset of M_1. If $B = \bigcap|{:}F{:}|$, then B is operations closed.

2. Relationships between Subsets Families

Let I be a set, let M be a many sorted set indexed by I, let B be a family of many sorted subsets indexed by M, and let A be a family of many sorted subsets indexed by M. We say that A is finer than B if and only if:

(Def. 1) For every set a such that $a \in A$ there exists a set b such that $b \in B$ and $a \subseteq b$.

Let us note that the predicate A is finer than B is reflexive. We say that B is coarser than A if and only if:

(Def. 2) For every set b such that $b \in B$ there exists a set a such that $a \in A$ and $a \subseteq b$.

Let us note that the predicate B is coarser than A is reflexive.

One can prove the following two propositions:

(4) Let I be a set, M be a many sorted set indexed by I, and A, B, C be families of many sorted subsets indexed by M. If A is finer than B and B is finer than C, then A is finer than C.

(5) Let I be a set, M be a many sorted set indexed by I, and A, B, C be families of many sorted subsets indexed by M. If A is coarser than B and B is coarser than C, then A is coarser than C.

Let I be a non empty set and let M be a many sorted set indexed by I. The functor $\operatorname{supp}(M)$ yields a set and is defined as follows:

(Def. 3) $\operatorname{supp}(M) = \{x; x$ ranges over elements of I: $M(x) \neq \emptyset\}$.

The following propositions are true:

(6) For every non empty set I and for every non-empty many sorted set M indexed by I holds $M = \mathbf{0}_I +\cdot M \restriction \operatorname{supp}(M)$.

(7) Let I be a non empty set and M_2, M_3 be non-empty many sorted sets indexed by I. If $\operatorname{supp}(M_2) = \operatorname{supp}(M_3)$ and $M_2 \restriction \operatorname{supp}(M_2) = M_3 \restriction \operatorname{supp}(M_3)$, then $M_2 = M_3$.

(8) Let I be a non empty set, M be a many sorted set indexed by I, and x be an element of I. If $x \notin \operatorname{supp}(M)$, then $M(x) = \emptyset$.

(9) Let I be a non empty set, M be a many sorted set indexed by I, x be an element of $\operatorname{Bool}(M)$, i be an element of I, and y be a set. Suppose $y \in x(i)$. Then there exists an element a of $\operatorname{Bool}(M)$ such that $y \in a(i)$ and a is locally-finite and $\operatorname{supp}(a)$ is finite and $a \subseteq x$.

Let I be a set, let M be a many sorted set indexed by I, and let A be a family of many sorted subsets indexed by M. The functor $\operatorname{MSUnion}(A)$ yielding a many sorted subset indexed by M is defined as follows:

(Def. 4) For every set i such that $i \in I$ holds $(\operatorname{MSUnion}(A))(i) = \bigcup\{f(i); f$ ranges over elements of $\operatorname{Bool}(M)$: $f \in A\}$.

Let I be a set, let M be a many sorted set indexed by I, and let B be a non empty family of many sorted subsets indexed by M. We see that the element of B is a many sorted set indexed by I.

Let I be a set, let M be a many sorted set indexed by I, and let A be an empty family of many sorted subsets indexed by M. Observe that $\operatorname{MSUnion}(A)$ is empty yielding.

The following proposition is true

(10) Let I be a set, M be a many sorted set indexed by I, and A be a family of many sorted subsets indexed by M. Then $\operatorname{MSUnion}(A) = \bigcup|{:}A{:}|$.

Let I be a set, let M be a many sorted set indexed by I, and let A, B be families of many sorted subsets indexed by M. Then $A \cup B$ is a family of many sorted subsets indexed by M.

The following two propositions are true:

(11) Let I be a set, M be a many sorted set indexed by I, and A, B be families of many sorted subsets indexed by M. Then $\operatorname{MSUnion}(A \cup B) = \operatorname{MSUnion}(A) \cup \operatorname{MSUnion}(B)$.

(12) Let I be a set, M be a many sorted set indexed by I, and A, B be families of many sorted subsets indexed by M. If $A \subseteq B$, then $\operatorname{MSUnion}(A) \subseteq \operatorname{MSUnion}(B)$.

Let I be a set, let M be a many sorted set indexed by I, and let A, B be families of many sorted subsets indexed by M. Then $A \cap B$ is a family of many sorted subsets indexed by M.

One can prove the following two propositions:

(13) Let I be a set, M be a many sorted set indexed by I, and A, B be families of many sorted subsets indexed by M. Then $\operatorname{MSUnion}(A \cap B) \subseteq \operatorname{MSUnion}(A) \cap \operatorname{MSUnion}(B)$.

(14) Let I be a set, M be a many sorted set indexed by I, and A_1 be a set. Suppose that for every set x such that $x \in A_1$ holds x is a family of many sorted subsets indexed by M. Let A, B be families of many sorted subsets indexed by M. Suppose $B = \{\operatorname{MSUnion}(X); X$ ranges over families of many sorted subsets indexed by M: $X \in A_1\}$ and $A = \bigcup A_1$. Then $\operatorname{MSUnion}(B) = \operatorname{MSUnion}(A)$.

3. Algebraic Operation on Subsets of Many Sorted Sets

Let I be a non empty set, let M be a many sorted set indexed by I, and let S be a set operation in M. We say that S is algebraic if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let x be an element of $\mathrm{Bool}(M)$. Suppose $x = S(x)$. Then there exists a family A of many sorted subsets indexed by M such that $A = \{S(a); a$ ranges over elements of $\mathrm{Bool}(M)$: a is locally-finite $\wedge$ $\mathrm{supp}(a)$is finite $\wedge$ $a \subseteq x\}$ and $x = \mathrm{MSUnion}(A)$.

Let I be a non empty set and let M be a many sorted set indexed by I. Note that there exists a set operation in M which is algebraic, reflexive, monotonic, and idempotent.

Let S be a non empty 1-sorted structure and let I_1 be a closure system of S. We say that I_1 is algebraic if and only if:

(Def. 6) $\mathrm{ClOp}(I_1)$ is algebraic.

Let S be a non void non empty many sorted signature and let M_1 be a non-empty algebra over S. The functor $\mathrm{SubAlgCl}(M_1)$ yielding a strict closure system structure over the 1-sorted structure of S is defined by:

(Def. 7) The sorts of $\mathrm{SubAlgCl}(M_1)$ = the sorts of M_1 and the family of $\mathrm{SubAlgCl}(M_1)$ = $\mathrm{SubSorts}(M_1)$.

We now state the proposition

(16)[1] Let S be a non void non empty many sorted signature and M_1 be a strict non-empty algebra over S. Then $\mathrm{SubSorts}(M_1)$ is an absolutely-multiplicative family of many sorted subsets indexed by the sorts of M_1.

Let S be a non void non empty many sorted signature and let M_1 be a strict non-empty algebra over S. One can check that $\mathrm{SubAlgCl}(M_1)$ is absolutely-multiplicative.

Let S be a non void non empty many sorted signature and let M_1 be a strict non-empty algebra over S. Note that $\mathrm{SubAlgCl}(M_1)$ is algebraic.

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Received June 23, 1997

Published January 2, 2004