

On the Closure Operator and the Closure System of Many Sorted Sets

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Summary. In this paper definitions of many sorted closure system and many sorted closure operator are introduced. These notations are also introduced in [9], but in another meaning. In this article closure system is absolutely multiplicative subset family of many sorted sets and in [9] is many sorted absolutely multiplicative subset family of many sorted sets. Analogously, closure operator is function between many sorted sets and in [9] is many sorted function from a many sorted set into a many sorted set.

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The articles [11], [5], [15], [10], [16], [2], [4], [3], [6], [12], [13], [14], [1], [8], and [7] provide the notation and terminology for this paper.

1. PRELIMINARIES

We adopt the following convention: i, x, I denote sets, A, B, M denote many sorted sets indexed by I , and f, f_1 denote functions.

Next we state two propositions:

- (1) For every non empty set M and for all elements X, Y of M such that $X \subseteq Y$ holds $\text{id}_M(X) \subseteq \text{id}_M(Y)$.
- (3)¹ Let I be a non empty set, A be a many sorted set indexed by I , and B be a many sorted subset indexed by A . Then $\text{rng} B \subseteq \bigcup \text{rng}(2^A)$.

Let us mention that every set which is empty is also functional.

Let us observe that there exists a set which is empty and functional.

Let f, g be functions. One can check that $\{f, g\}$ is functional.

2. SET OF MANY SORTED SUBSETS OF A MANY SORTED SET

Let us consider I, M . The functor $\text{Bool}(M)$ yielding a set is defined by:

(Def. 1) $x \in \text{Bool}(M)$ iff x is a many sorted subset indexed by M .

¹ The proposition (2) has been removed.

Let us consider I, M . Observe that $\text{Bool}(M)$ is non empty and functional and has common domain.

Let us consider I, M . A family of many sorted subsets indexed by M is a subset of $\text{Bool}(M)$.

Let us consider I, M . Then $\text{Bool}(M)$ is a family of many sorted subsets indexed by M .

Let us consider I, M . One can verify that there exists a family of many sorted subsets indexed by M which is non empty and functional and has common domain.

Let us consider I, M . Observe that there exists a family of many sorted subsets indexed by M which is empty and finite.

In the sequel S_1, S_2 denote families of many sorted subsets indexed by M .

Let us consider I, M and let S be a non empty family of many sorted subsets indexed by M . We see that the element of S is a many sorted subset indexed by M .

The following propositions are true:

- (4) $S_1 \cup S_2$ is a family of many sorted subsets indexed by M .
- (5) $S_1 \cap S_2$ is a family of many sorted subsets indexed by M .
- (6) $S_1 \setminus x$ is a family of many sorted subsets indexed by M .
- (7) $S_1 \dot{-} S_2$ is a family of many sorted subsets indexed by M .
- (8) If $A \subseteq M$, then $\{A\}$ is a family of many sorted subsets indexed by M .
- (9) If $A \subseteq M$ and $B \subseteq M$, then $\{A, B\}$ is a family of many sorted subsets indexed by M .

In the sequel E, T denote elements of $\text{Bool}(M)$.

We now state four propositions:

- (10) $E \cap T \in \text{Bool}(M)$.
- (11) $E \cup T \in \text{Bool}(M)$.
- (12) $E \setminus A \in \text{Bool}(M)$.
- (13) $E \dot{-} T \in \text{Bool}(M)$.

3. MANY SORTED OPERATOR CORRESPONDING TO THE OPERATOR ON MANY SORTED SUBSETS

Let S be a functional set. The functor $|S|$ yields a function and is defined as follows:

- (Def. 3)²(i) There exists a non empty functional set A such that $A = S$ and $\text{dom } |S| = \bigcap \{ \text{dom } x : x \text{ ranges over elements of } A \}$ and for every i such that $i \in \text{dom } |S|$ holds $|S|(i) = \{x(i) : x \text{ ranges over elements of } A\}$ if $S \neq \emptyset$,
- (ii) $|S| = \emptyset$, otherwise.

We now state the proposition

- (14) For every non empty family S_1 of many sorted subsets indexed by M holds $\text{dom } |S_1| = I$.

Let S be an empty functional set. Observe that $|S|$ is empty.

Let us consider I, M and let S be a family of many sorted subsets indexed by M . The functor $|\cdot S|$ yielding a many sorted set indexed by I is defined by:

- (Def. 4) $|\cdot S| = \begin{cases} |S|, & \text{if } S \neq \emptyset, \\ \mathbf{0}_I, & \text{otherwise.} \end{cases}$

Let us consider I, M and let S be an empty family of many sorted subsets indexed by M . One can verify that $|\cdot S|$ is empty yielding.

Next we state the proposition

² The definition (Def. 2) has been removed.

- (15) If S_1 is non empty, then for every i such that $i \in I$ holds $|\mathcal{S}_1:|(i) = \{x(i); x \text{ ranges over elements of } \text{Bool}(M): x \in S_1\}$.

Let us consider I, M and let S_1 be a non empty family of many sorted subsets indexed by M . One can check that $|\mathcal{S}_1:|$ is non-empty.

We now state several propositions:

- (16) $\text{dom}|\{f\}| = \text{dom}f$.
 (17) $\text{dom}|\{f, f_1\}| = \text{dom}f \cap \text{dom}f_1$.
 (18) If $i \in \text{dom}f$, then $|\{f\}|(i) = \{f(i)\}$.
 (19) If $i \in I$ and $S_1 = \{f\}$, then $|\mathcal{S}_1:|(i) = \{f(i)\}$.
 (20) If $i \in \text{dom}|\{f, f_1\}|$, then $|\{f, f_1\}|(i) = \{f(i), f_1(i)\}$.
 (21) If $i \in I$ and $S_1 = \{f, f_1\}$, then $|\mathcal{S}_1:|(i) = \{f(i), f_1(i)\}$.

Let us consider I, M, S_1 . Then $|\mathcal{S}_1:|$ is a subset family of M .

One can prove the following propositions:

- (22) If $A \in S_1$, then $A \in |\mathcal{S}_1:|$.
 (23) If $S_1 = \{A, B\}$, then $\bigcup|\mathcal{S}_1:| = A \cup B$.
 (24) If $S_1 = \{E, T\}$, then $\bigcap|\mathcal{S}_1:| = E \cap T$.
 (25) Let Z be a many sorted subset indexed by M . Suppose that for every many sorted set Z_1 indexed by I such that $Z_1 \in S_1$ holds $Z \subseteq Z_1$. Then $Z \subseteq \bigcap|\mathcal{S}_1:|$.
 (26) $|\text{Bool}(M):| = 2^M$.

Let us consider I, M and let I_1 be a family of many sorted subsets indexed by M . We say that I_1 is additive if and only if:

- (Def. 5) For all A, B such that $A \in I_1$ and $B \in I_1$ holds $A \cup B \in I_1$.

We say that I_1 is absolutely-additive if and only if:

- (Def. 6) For every family F of many sorted subsets indexed by M such that $F \subseteq I_1$ holds $\bigcup|F:| \in I_1$.

We say that I_1 is multiplicative if and only if:

- (Def. 7) For all A, B such that $A \in I_1$ and $B \in I_1$ holds $A \cap B \in I_1$.

We say that I_1 is absolutely-multiplicative if and only if:

- (Def. 8) For every family F of many sorted subsets indexed by M such that $F \subseteq I_1$ holds $\bigcap|F:| \in I_1$.

We say that I_1 is properly upper bound if and only if:

- (Def. 9) $M \in I_1$.

We say that I_1 is properly lower bound if and only if:

- (Def. 10) $\mathbf{0}_I \in I_1$.

Let us consider I, M . One can check that there exists a family of many sorted subsets indexed by M which is non empty, functional, additive, absolutely-additive, multiplicative, absolutely-multiplicative, properly upper bound, and properly lower bound and has common domain.

Let us consider I, M . Then $\text{Bool}(M)$ is an additive absolutely-additive multiplicative absolutely-multiplicative properly upper bound properly lower bound family of many sorted subsets indexed by M .

Let us consider I, M . Observe that every family of many sorted subsets indexed by M which is absolutely-additive is also additive.

Let us consider I, M . Observe that every family of many sorted subsets indexed by M which is absolutely-multiplicative is also multiplicative.

Let us consider I, M . One can check that every family of many sorted subsets indexed by M which is absolutely-multiplicative is also properly upper bound.

Let us consider I, M . Observe that every family of many sorted subsets indexed by M which is properly upper bound is also non empty.

Let us consider I, M . Observe that every family of many sorted subsets indexed by M which is absolutely-additive is also properly lower bound.

Let us consider I, M . One can verify that every family of many sorted subsets indexed by M which is properly lower bound is also non empty.

4. PROPERTIES OF CLOSURE OPERATORS

Let us consider I, M . A set operation in M is a function from $\text{Bool}(M)$ into $\text{Bool}(M)$.

Let us consider I, M , let f be a set operation in M , and let x be an element of $\text{Bool}(M)$. Then $f(x)$ is an element of $\text{Bool}(M)$.

Let us consider I, M and let I_1 be a set operation in M . We say that I_1 is reflexive if and only if:

(Def. 12)³ For every element x of $\text{Bool}(M)$ holds $x \subseteq I_1(x)$.

We say that I_1 is monotonic if and only if:

(Def. 13) For all elements x, y of $\text{Bool}(M)$ such that $x \subseteq y$ holds $I_1(x) \subseteq I_1(y)$.

We say that I_1 is idempotent if and only if:

(Def. 14) For every element x of $\text{Bool}(M)$ holds $I_1(x) = I_1(I_1(x))$.

We say that I_1 is topological if and only if:

(Def. 15) For all elements x, y of $\text{Bool}(M)$ holds $I_1(x \cup y) = I_1(x) \cup I_1(y)$.

Let us consider I, M . Note that there exists a set operation in M which is reflexive, monotonic, idempotent, and topological.

The following propositions are true:

- (27) $\text{id}_{\text{Bool}(A)}$ is a reflexive set operation in A .
- (28) $\text{id}_{\text{Bool}(A)}$ is a monotonic set operation in A .
- (29) $\text{id}_{\text{Bool}(A)}$ is an idempotent set operation in A .
- (30) $\text{id}_{\text{Bool}(A)}$ is a topological set operation in A .

In the sequel g, h are set operations in M .

We now state three propositions:

- (31) If $E = M$ and g is reflexive, then $E = g(E)$.
- (32) If g is reflexive and for every element X of $\text{Bool}(M)$ holds $g(X) \subseteq X$, then g is idempotent.
- (33) For every element A of $\text{Bool}(M)$ such that $A = E \cap T$ holds if g is monotonic, then $g(A) \subseteq g(E) \cap g(T)$.

Let us consider I, M . Note that every set operation in M which is topological is also monotonic. One can prove the following proposition

³ The definition (Def. 11) has been removed.

- (34) For every element A of $\text{Bool}(M)$ such that $A = E \setminus T$ holds if g is topological, then $g(E) \setminus g(T) \subseteq g(A)$.

Let us consider I, M, h, g . Then $g \cdot h$ is a set operation in M .

Next we state four propositions:

- (35) If g is reflexive and h is reflexive, then $g \cdot h$ is reflexive.
 (36) If g is monotonic and h is monotonic, then $g \cdot h$ is monotonic.
 (37) If g is idempotent and h is idempotent and $g \cdot h = h \cdot g$, then $g \cdot h$ is idempotent.
 (38) If g is topological and h is topological, then $g \cdot h$ is topological.

5. ON THE CLOSURE OPERATOR AND THE CLOSURE SYSTEM

In the sequel S is a 1-sorted structure.

Let us consider S . We introduce closure system structures over S which are extensions of many-sorted structure over S and are systems

$\langle \text{sorts, a family} \rangle$,

where the sorts constitute a many sorted set indexed by the carrier of S and the family is a family of many sorted subsets indexed by the sorts.

In the sequel M_1 denotes a many-sorted structure over S .

Let us consider S and let I_1 be a closure system structure over S . We say that I_1 is additive if and only if:

- (Def. 16) The family of I_1 is additive.

We say that I_1 is absolutely-additive if and only if:

- (Def. 17) The family of I_1 is absolutely-additive.

We say that I_1 is multiplicative if and only if:

- (Def. 18) The family of I_1 is multiplicative.

We say that I_1 is absolutely-multiplicative if and only if:

- (Def. 19) The family of I_1 is absolutely-multiplicative.

We say that I_1 is properly upper bound if and only if:

- (Def. 20) The family of I_1 is properly upper bound.

We say that I_1 is properly lower bound if and only if:

- (Def. 21) The family of I_1 is properly lower bound.

Let us consider S, M_1 . The functor $\text{Full}(M_1)$ yielding a closure system structure over S is defined as follows:

- (Def. 22) $\text{Full}(M_1) = \langle \text{the sorts of } M_1, \text{Bool}(\text{the sorts of } M_1) \rangle$.

Let us consider S, M_1 . Note that $\text{Full}(M_1)$ is strict, additive, absolutely-additive, multiplicative, absolutely-multiplicative, properly upper bound, and properly lower bound.

Let us consider S and let M_1 be a non-empty many-sorted structure over S . Observe that $\text{Full}(M_1)$ is non-empty.

Let us consider S . Note that there exists a closure system structure over S which is strict, non-empty, additive, absolutely-additive, multiplicative, absolutely-multiplicative, properly upper bound, and properly lower bound.

Let us consider S and let C_1 be an additive closure system structure over S . Note that the family of C_1 is additive.

Let us consider S and let C_1 be an absolutely-additive closure system structure over S . Note that the family of C_1 is absolutely-additive.

Let us consider S and let C_1 be a multiplicative closure system structure over S . Note that the family of C_1 is multiplicative.

Let us consider S and let C_1 be an absolutely-multiplicative closure system structure over S . One can verify that the family of C_1 is absolutely-multiplicative.

Let us consider S and let C_1 be a properly upper bound closure system structure over S . One can check that the family of C_1 is properly upper bound.

Let us consider S and let C_1 be a properly lower bound closure system structure over S . Note that the family of C_1 is properly lower bound.

Let us consider S , let M be a non-empty many sorted set indexed by the carrier of S , and let F be a family of many sorted subsets indexed by M . One can check that $\langle M, F \rangle$ is non-empty.

Let us consider S , M_1 and let F be an additive family of many sorted subsets indexed by the sorts of M_1 . Note that \langle the sorts of $M_1, F \rangle$ is additive.

Let us consider S , M_1 and let F be an absolutely-additive family of many sorted subsets indexed by the sorts of M_1 . Observe that \langle the sorts of $M_1, F \rangle$ is absolutely-additive.

Let us consider S , M_1 and let F be a multiplicative family of many sorted subsets indexed by the sorts of M_1 . One can check that \langle the sorts of $M_1, F \rangle$ is multiplicative.

Let us consider S , M_1 and let F be an absolutely-multiplicative family of many sorted subsets indexed by the sorts of M_1 . Observe that \langle the sorts of $M_1, F \rangle$ is absolutely-multiplicative.

Let us consider S , M_1 and let F be a properly upper bound family of many sorted subsets indexed by the sorts of M_1 . One can check that \langle the sorts of $M_1, F \rangle$ is properly upper bound.

Let us consider S , M_1 and let F be a properly lower bound family of many sorted subsets indexed by the sorts of M_1 . Observe that \langle the sorts of $M_1, F \rangle$ is properly lower bound.

Let us consider S . One can check that every closure system structure over S which is absolutely-additive is also additive.

Let us consider S . Observe that every closure system structure over S which is absolutely-multiplicative is also multiplicative.

Let us consider S . Note that every closure system structure over S which is absolutely-multiplicative is also properly upper bound.

Let us consider S . One can verify that every closure system structure over S which is absolutely-additive is also properly lower bound.

Let us consider S . A closure system of S is an absolutely-multiplicative closure system structure over S .

Let us consider I, M . A closure operator of M is a reflexive monotonic idempotent set operation in M .

Next we state the proposition

- (39) Let A be a many sorted set indexed by the carrier of S , f be a reflexive monotonic set operation in A , and D be a family of many sorted subsets indexed by A . Suppose $D = \{x; x \text{ ranges over elements of } \text{Bool}(A): f(x) = x\}$. Then $\langle A, D \rangle$ is a closure system of S .

Let us consider S , let A be a many sorted set indexed by the carrier of S , and let g be a closure operator of A . The functor $\text{ClSys}(g)$ yields a strict closure system of S and is defined by:

(Def. 23) The sorts of $\text{ClSys}(g) = A$ and the family of $\text{ClSys}(g) = \{x; x \text{ ranges over elements of } \text{Bool}(A): g(x) = x\}$.

Let us consider S , let A be a closure system of S , and let C be a many sorted subset indexed by the sorts of A . The functor \bar{C} yields an element of $\text{Bool}(\text{the sorts of } A)$ and is defined by the condition (Def. 24).

(Def. 24) There exists a family F of many sorted subsets indexed by the sorts of A such that $\bar{C} = \bigcap \{F; F = \{X; X \text{ ranges over elements of } \text{Bool}(\text{the sorts of } A): C \subseteq X \wedge X \in \text{the family of } A\}$.

The following propositions are true:

- (40) Let D be a closure system of S , a be an element of $\text{Bool}(\text{the sorts of } D)$, and f be a set operation in the sorts of D . Suppose $a \in$ the family of D and for every element x of $\text{Bool}(\text{the sorts of } D)$ holds $f(x) = \bar{x}$. Then $f(a) = a$.
- (41) Let D be a closure system of S , a be an element of $\text{Bool}(\text{the sorts of } D)$, and f be a set operation in the sorts of D . Suppose $f(a) = a$ and for every element x of $\text{Bool}(\text{the sorts of } D)$ holds $f(x) = \bar{x}$. Then $a \in$ the family of D .
- (42) Let D be a closure system of S and f be a set operation in the sorts of D . Suppose that for every element x of $\text{Bool}(\text{the sorts of } D)$ holds $f(x) = \bar{x}$. Then f is reflexive, monotonic, and idempotent.

Let us consider S and let D be a closure system of S . The functor $\text{ClOp}(D)$ yields a closure operator of the sorts of D and is defined as follows:

(Def. 25) For every element x of $\text{Bool}(\text{the sorts of } D)$ holds $(\text{ClOp}(D))(x) = \bar{x}$.

Next we state two propositions:

- (43) For every many sorted set A indexed by the carrier of S and for every closure operator f of A holds $\text{ClOp}(\text{ClSys}(f)) = f$.
- (44) For every closure system D of S holds $\text{ClSys}(\text{ClOp}(D)) =$ the closure system structure of D .

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