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Universal Classes

Bogdan Nowak
Łódź University

Grzegorz Bancerek
Warsaw University
Białystok

Summary. In the article we have shown that there exist universal classes, i.e. there are sets which are closed w.r.t. basic set theory operations.

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The articles [12], [8], [13], [4], [9], [11], [14], [6], [7], [2], [3], [1], [5], and [10] provide the notation and terminology for this paper.

We use the following convention: m is a cardinal number, A, B, C are ordinal numbers, and x, y, X, Y, W are sets.

Let us note that every set which is a Tarski class is also subset-closed.

Let X be a set. Observe that $\mathbf{T}(X)$ is a Tarski class.

We now state four propositions:

(1) If W is subset-closed and $X \in W$, then $X \not\approx W$ and $\overline{\overline{X}} < \overline{\overline{W}}$.

(3)[1] If W is a Tarski class and $x \in W$ and $y \in W$, then $\{x\} \in W$ and $\{x,y\} \in W$.

(4) If W is a Tarski class and $x \in W$ and $y \in W$, then $\langle x, y \rangle \in W$.

(5) If W is a Tarski class and $X \in W$, then $\mathbf{T}(X) \subseteq W$.

The scheme *TC* concerns a unary predicate $\mathcal{P}$, and states that:

For every X holds $\mathcal{P}[\mathbf{T}(X)]$

provided the following requirement is met:

- For every X such that X is a Tarski class holds $\mathcal{P}[X]$.

One can prove the following propositions:

(6) If W is a Tarski class and $A \in W$, then $\mathrm{succ}A \in W$ and $A \subseteq W$.

(7) If $A \in \mathbf{T}(W)$, then $\mathrm{succ}A \in \mathbf{T}(W)$ and $A \subseteq \mathbf{T}(W)$.

(8) If W is subset-closed and X is transitive and $X \in W$, then $X \subseteq W$.

(9) If X is transitive and $X \in \mathbf{T}(W)$, then $X \subseteq \mathbf{T}(W)$.

(10) If W is a Tarski class, then $\mathrm{On}\,W = \overline{\overline{W}}$.

(11) $\mathrm{On}\,\mathbf{T}(W) = \overline{\overline{\mathbf{T}(W)}}$.

(12) If W is a Tarski class and $X \in W$, then $\overline{\overline{X}} \in W$.

[1] The proposition (2) has been removed.

(13) If $X \in \mathbf{T}(W)$, then $\overline{\overline{X}} \in \mathbf{T}(W)$.

(14) If W is a Tarski class and $x \in \overline{\overline{W}}$, then $x \in W$.

(15) If $x \in \overline{\overline{\mathbf{T}(W)}}$, then $x \in \mathbf{T}(W)$.

(16) If W is a Tarski class and $m < \overline{\overline{W}}$, then $m \in W$.

(17) If $m < \overline{\overline{\mathbf{T}(W)}}$, then $m \in \mathbf{T}(W)$.

(18) If W is a Tarski class and $m \in W$, then $m \subseteq W$.

(19) If $m \in \mathbf{T}(W)$, then $m \subseteq \mathbf{T}(W)$.

(20) If W is a Tarski class, then $\overline{\overline{W}}$ is a limit ordinal number.

(21) If W is a Tarski class and $W \neq \emptyset$, then $\overline{\overline{W}} \neq 0$ and $\overline{\overline{W}} \neq \emptyset$ and $\overline{\overline{W}}$ is a limit ordinal number.

(22) $\overline{\overline{\mathbf{T}(W)}} \neq 0$ and $\overline{\overline{\mathbf{T}(W)}} \neq \emptyset$ and $\overline{\overline{\mathbf{T}(W)}}$ is a limit ordinal number.

In the sequel L is a transfinite sequence.
Next we state a number of propositions:

(23) If W is a Tarski class and if $X \in W$ and W is transitive or $X \in W$ and $X \subseteq W$ or $\overline{\overline{X}} < \overline{\overline{W}}$ and $X \subseteq W$, then $W^X \subseteq W$.

(24) If $X \in \mathbf{T}(W)$ and W is transitive or $X \in \mathbf{T}(W)$ and $X \subseteq \mathbf{T}(W)$ or $\overline{\overline{X}} < \overline{\overline{\mathbf{T}(W)}}$ and $X \subseteq \mathbf{T}(W)$, then $\mathbf{T}(W)^X \subseteq \mathbf{T}(W)$.

(25) If $\mathrm{dom} L$ is a limit ordinal number and for every A such that $A \in \mathrm{dom} L$ holds $L(A) = \mathbf{R}_A$, then $\mathbf{R}_{\mathrm{dom} L} = \bigcup L$.

(26) If W is a Tarski class and $A \in \mathrm{On}\, W$, then $\overline{\overline{\mathbf{R}_A}} < \overline{\overline{W}}$ and $\mathbf{R}_A \in W$.

(27) If $A \in \mathrm{On}\, \mathbf{T}(W)$, then $\overline{\overline{\mathbf{R}_A}} < \overline{\overline{\mathbf{T}(W)}}$ and $\mathbf{R}_A \in \mathbf{T}(W)$.

(28) If W is a Tarski class, then $\mathbf{R}_{\overline{\overline{W}}} \subseteq W$.

(29) $\mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}} \subseteq \mathbf{T}(W)$.

(30) If W is a Tarski class and transitive and $X \in W$, then $\mathrm{rk}(X) \in W$.

(31) If W is a Tarski class and transitive, then $W \subseteq \mathbf{R}_{\overline{\overline{W}}}$.

(32) If W is a Tarski class and transitive, then $\mathbf{R}_{\overline{\overline{W}}} = W$.

(33) If W is a Tarski class and $A \in \mathrm{On}\, W$, then $\overline{\overline{\mathbf{R}_A}} \leq \overline{\overline{W}}$.

(34) If $A \in \mathrm{On}\, \mathbf{T}(W)$, then $\overline{\overline{\mathbf{R}_A}} \leq \overline{\overline{\mathbf{T}(W)}}$.

(35) If W is a Tarski class, then $\overline{\overline{W}} = \overline{\overline{\mathbf{R}_{\overline{\overline{W}}}}}$.

(36) $\overline{\overline{\mathbf{T}(W)}} = \overline{\overline{\mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}}}}$.

(37) If W is a Tarski class and $X \subseteq \mathbf{R}_{\overline{\overline{W}}}$, then $X \approx \mathbf{R}_{\overline{\overline{W}}}$ or $X \in \mathbf{R}_{\overline{\overline{W}}}$.

(38) If $X \subseteq \mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}}$, then $X \approx \mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}}$ or $X \in \mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}}$.

(39) If W is a Tarski class, then $\mathbf{R}_{\overline{\overline{W}}}$ is a Tarski class.

(40) $\mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}}$ is a Tarski class.

(41) If X is transitive and $A \in \mathrm{rk}(X)$, then there exists Y such that $Y \in X$ and $\mathrm{rk}(Y) = A$.

(42) If X is transitive, then $\overline{\overline{\mathrm{rk}(X)}} \leq \overline{\overline{X}}$.

(43) If W is a Tarski class and X is transitive and $X \in W$, then $X \in \mathbf{R}_{\overline{\overline{W}}}$.

(44) If X is transitive and $X \in \mathbf{T}(W)$, then $X \in \mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}}$.

(45) If W is transitive, then $\mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}}$ is a Tarski class of W.

(46) If W is transitive, then $\mathbf{R}_{\overline{\overline{\mathbf{T}(W)}}} = \mathbf{T}(W)$.

Let I_1 be a set. We say that I_1 is universal if and only if:

(Def. 1) I_1 is transitive and a Tarski class.

Let us mention that every set which is universal is also transitive and a Tarski class and every set which is transitive and a Tarski class is also universal.

Let us mention that there exists a set which is universal and non empty.

A universal class is a universal non empty set.

In the sequel U_1, U_2, U_3, U_4 are universal classes.

We now state three propositions:

(50)[2] $\mathrm{On}\,U_4$ is an ordinal number.

(51) If X is transitive, then $\mathbf{T}(X)$ is universal.

(52) $\mathbf{T}(U_4)$ is a universal class.

Let us consider U_4. One can check that $\mathrm{On}\,U_4$ is ordinal and $\mathbf{T}(U_4)$ is universal.

We now state the proposition

(53) $\mathbf{T}(A)$ is universal.

Let us consider A. Observe that $\mathbf{T}(A)$ is universal.

The following propositions are true:

(54) $U_4 = \mathbf{R}_{\mathrm{On}\,U_4}$.

(55) $\mathrm{On}\,U_4 \neq \emptyset$ and $\mathrm{On}\,U_4$ is a limit ordinal number.

(56) $U_1 \in U_2$ or $U_1 = U_2$ or $U_2 \in U_1$.

(57) $U_1 \subseteq U_2$ or $U_2 \in U_1$.

(58) U_1 and U_2 are $\subseteq$-comparable.

(59) If $U_1 \in U_2$ and $U_2 \in U_3$, then $U_1 \in U_3$.

(61)[3] $U_1 \cup U_2$ is a universal class and $U_1 \cap U_2$ is a universal class.

(62) $\emptyset \in U_4$.

(63) If $x \in U_4$, then $\{x\} \in U_4$.

(64) If $x \in U_4$ and $y \in U_4$, then $\{x, y\} \in U_4$ and $\langle x, y \rangle \in U_4$.

(65) If $X \in U_4$, then $2^X \in U_4$ and $\bigcup X \in U_4$ and $\bigcap X \in U_4$.

(66) If $X \in U_4$ and $Y \in U_4$, then $X \cup Y \in U_4$ and $X \cap Y \in U_4$ and $X \setminus Y \in U_4$ and $X \dot{-} Y \in U_4$.

[2] The propositions (47)–(49) have been removed.

[3] The proposition (60) has been removed.

(67) If $X \in U_4$ and $Y \in U_4$, then $[:X, Y:] \in U_4$ and $Y^X \in U_4$.

In the sequel u, v denote elements of U_4.

Let us consider U_1. One can verify that there exists an element of U_1 which is non empty.

Let us consider U_4, u. Then $\{u\}$ is an element of U_4. Then 2^u is a non empty element of U_4. Then $\bigcup u$ is an element of U_4. Then $\bigcap u$ is an element of U_4. Let us consider v. Then $\{u,v\}$ is an element of U_4. Then $\langle u, v\rangle$ is an element of U_4. Then $u \cup v$ is an element of U_4. Then $u \cap v$ is an element of U_4. Then $u \setminus v$ is an element of U_4. Then $u \dot{-} v$ is an element of U_4. Then $[:u, v:]$ is an element of U_4. Then v^u is an element of U_4.

The universal class $\mathbf{U}_0$ is defined as follows:

(Def. 2) $\mathbf{U}_0 = \mathbf{T}(\emptyset)$.

One can prove the following three propositions:

(69)[4] $\overline{\overline{\mathbf{R}_\omega}} = \overline{\overline{\omega}}$.

(70) $\mathbf{R}_\omega$ is a Tarski class.

(71) $\mathbf{U}_0 = \mathbf{R}_\omega$.

The universal class $\mathbf{U}_1$ is defined as follows:

(Def. 3) $\mathbf{U}_1 = \mathbf{T}(\mathbf{U}_0)$.

Let X be a set. One can verify that $X^{*\in}$ is transitive.

Let X be a transitive set. Observe that $\mathbf{T}(X)$ is transitive.

Let A be an ordinal number. Observe that $\mathbf{R}_A$ is transitive.

Let X be a set. The functor Universe_closure(X) yielding a universal class is defined by:

(Def. 4) $X \subseteq$ Universe_closure(X) and for every universal class Y such that $X \subseteq Y$ holds Universe_closure$(X) \subseteq Y$.

A set of a finite rank is an element of $\mathbf{U}_0$. A *Set* is an element of $\mathbf{U}_1$. Let us consider A. The functor $\mathbf{U}_A$ is defined by the condition (Def. 5).

(Def. 5) There exists L such that

(i) $\mathbf{U}_A = \mathrm{last}\, L$,

(ii) $\mathrm{dom}\, L = \mathrm{succ}\, A$,

(iii) $L(\emptyset) = \mathbf{U}_0$,

(iv) for every C such that $\mathrm{succ}\, C \in \mathrm{succ}\, A$ holds $L(\mathrm{succ}\, C) = \mathbf{T}(L(C))$, and

(v) for every C such that $C \in \mathrm{succ}\, A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L(C) =$ Universe_closure$(\bigcup(L{\restriction}C))$.

Let us consider A. One can verify that $\mathbf{U}_A$ is universal and non empty.

One can prove the following propositions:

(75)[5] $\mathbf{U}_\emptyset = \mathbf{U}_0$.

(76) $\mathbf{U}_{\mathrm{succ}\, A} = \mathbf{T}(\mathbf{U}_A)$.

(77) $\mathbf{U}_{\mathbf{1}} = \mathbf{U}_1$.

(78) If $A \neq \emptyset$ and A is a limit ordinal number and $\mathrm{dom}\, L = A$ and for every B such that $B \in A$ holds $L(B) = \mathbf{U}_B$, then $\mathbf{U}_A =$ Universe_closure$(\bigcup L)$.

(79) $\mathbf{U}_0 \subseteq U_4$ and $\mathbf{T}(\emptyset) \subseteq U_4$ and $\mathbf{U}_\emptyset \subseteq U_4$.

(80) $A \in B$ iff $\mathbf{U}_A \in \mathbf{U}_B$.

(81) If $\mathbf{U}_A = \mathbf{U}_B$, then $A = B$.

(82) $A \subseteq B$ iff $\mathbf{U}_A \subseteq \mathbf{U}_B$.

[4] The proposition (68) has been removed.

[5] The propositions (72)–(74) have been removed.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/card_1.html.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[3] Grzegorz Bancerek. Sequences of ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal2.html.

[4] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/wellord2.html.

[5] Grzegorz Bancerek. Tarski's classes and ranks. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/classes1.html.

[6] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[7] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[8] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[9] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[10] Andrzej Nędzusiak. σ-fields and probability. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/prob_1.html.

[11] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setfam_1.html.

[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[13] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[14] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

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