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Tarski's Classes and Ranks

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Summary. In the article the Tarski's classes (non-empty families of sets satisfying Tarski's axiom A given in [7]) and the rank sets are introduced and some of their properties are shown. The transitive closure and the rank of a set is given here too.

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The articles [7], [6], [9], [10], [5], [8], [2], [3], [4], and [1] provide the notation and terminology for this paper.

We adopt the following rules: W, X, Y, Z are sets, f is a function, and x, y are sets.

Let B be a set. We say that B is subset-closed if and only if:

(Def. 1) For all X, Y such that $X \in B$ and $Y \subseteq X$ holds $Y \in B$.

Let B be a set. We say that B is a Tarski class if and only if:

(Def. 2) B is subset-closed and for every X such that $X \in B$ holds $2^X \in B$ and for every X such that $X \subseteq B$ holds $X \approx B$ or $X \in B$.

We introduce B is a Tarski class as a synonym of B is a Tarski class.

Let A, B be sets. We say that B is a Tarski class of A if and only if:

(Def. 3) $A \in B$ and B is a Tarski class.

Let A be a set. The functor $\mathbf{T}(A)$ yielding a set is defined as follows:

(Def. 4) $\mathbf{T}(A)$ is a Tarski class of A and for every set D such that D is a Tarski class of A holds $\mathbf{T}(A) \subseteq D$.

Let A be a set. Note that $\mathbf{T}(A)$ is non empty.

The following propositions are true:

(2)[1] W is a Tarski class if and only if the following conditions are satisfied:

(i) W is subset-closed,

(ii) for every X such that $X \in W$ holds $2^X \in W$, and

(iii) for every X such that $X \subseteq W$ and $\overline{\overline{X}} < \overline{\overline{W}}$ holds $X \in W$.

(5)[2] $X \in \mathbf{T}(X)$.

(6) If $Y \in \mathbf{T}(X)$ and $Z \subseteq Y$, then $Z \in \mathbf{T}(X)$.

[1] The proposition (1) has been removed.
[2] The propositions (3) and (4) have been removed.

(7) If $Y \in \mathbf{T}(X)$, then $2^Y \in \mathbf{T}(X)$.

(8) If $Y \subseteq \mathbf{T}(X)$, then $Y \approx \mathbf{T}(X)$ or $Y \in \mathbf{T}(X)$.

(9) If $Y \subseteq \mathbf{T}(X)$ and $\overline{\overline{Y}} < \overline{\overline{\mathbf{T}(X)}}$, then $Y \in \mathbf{T}(X)$.

We use the following convention: u, v are elements of $\mathbf{T}(X)$, A, B, C are ordinal numbers, and L is a transfinite sequence.

Let us consider X, A. The functor $\mathbf{T}_A(X)$ is defined by the condition (Def. 5).

(Def. 5) There exists L such that

(i) $\mathbf{T}_A(X) = \mathrm{last}\, L$,

(ii) $\mathrm{dom}\, L = \mathrm{succ}\, A$,

(iii) $L(\emptyset) = \{X\}$,

(iv) for every C such that $\mathrm{succ}\, C \in \mathrm{succ}\, A$ holds $L(\mathrm{succ}\, C) = \{u : \bigvee_v (v \in L(C) \wedge u \subseteq v)\} \cup \{2^v : v \in L(C)\} \cup 2^{L(C)} \cap \mathbf{T}(X)$, and

(v) for every C such that $C \in \mathrm{succ}\, A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L(C) = \bigcup \mathrm{rng}(L \restriction C) \cap \mathbf{T}(X)$.

Let us consider X, A. Then $\mathbf{T}_A(X)$ is a subset of $\mathbf{T}(X)$.

One can prove the following propositions:

(10) $\mathbf{T}_{\emptyset}(X) = \{X\}$.

(11) $\mathbf{T}_{\mathrm{succ}\, A}(X) = \{u : \bigvee_v (v \in \mathbf{T}_A(X) \wedge u \subseteq v)\} \cup \{2^v : v \in \mathbf{T}_A(X)\} \cup 2^{\mathbf{T}_A(X)} \cap \mathbf{T}(X)$.

(12) If $A \neq \emptyset$ and A is a limit ordinal number, then $\mathbf{T}_A(X) = \{u : \bigvee_B (B \in A \wedge u \in \mathbf{T}_B(X))\}$.

(13) $Y \in \mathbf{T}_{\mathrm{succ}\, A}(X)$ iff $Y \subseteq \mathbf{T}_A(X)$ and $Y \in \mathbf{T}(X)$ or there exists Z such that $Z \in \mathbf{T}_A(X)$ but $Y \subseteq Z$ or $Y = 2^Z$.

(14) If $Y \subseteq Z$ and $Z \in \mathbf{T}_A(X)$, then $Y \in \mathbf{T}_{\mathrm{succ}\, A}(X)$.

(15) If $Y \in \mathbf{T}_A(X)$, then $2^Y \in \mathbf{T}_{\mathrm{succ}\, A}(X)$.

(16) If $A \neq \emptyset$ and A is a limit ordinal number, then $x \in \mathbf{T}_A(X)$ iff there exists B such that $B \in A$ and $x \in \mathbf{T}_B(X)$.

(17) If $A \neq \emptyset$ and A is a limit ordinal number and $Y \in \mathbf{T}_A(X)$ and $Z \subseteq Y$ or $Z = 2^Y$, then $Z \in \mathbf{T}_A(X)$.

(18) $\mathbf{T}_A(X) \subseteq \mathbf{T}_{\mathrm{succ}\, A}(X)$.

(19) If $A \subseteq B$, then $\mathbf{T}_A(X) \subseteq \mathbf{T}_B(X)$.

(20) There exists A such that $\mathbf{T}_A(X) = \mathbf{T}_{\mathrm{succ}\, A}(X)$.

(21) If $\mathbf{T}_A(X) = \mathbf{T}_{\mathrm{succ}\, A}(X)$, then $\mathbf{T}_A(X) = \mathbf{T}(X)$.

(22) There exists A such that $\mathbf{T}_A(X) = \mathbf{T}(X)$.

(23) There exists A such that $\mathbf{T}_A(X) = \mathbf{T}(X)$ and for every B such that $B \in A$ holds $\mathbf{T}_B(X) \neq \mathbf{T}(X)$.

(24) If $Y \neq X$ and $Y \in \mathbf{T}(X)$, then there exists A such that $Y \notin \mathbf{T}_A(X)$ and $Y \in \mathbf{T}_{\mathrm{succ}\, A}(X)$.

(25) If X is transitive, then for every A such that $A \neq \emptyset$ holds $\mathbf{T}_A(X)$ is transitive.

(26) $\mathbf{T}_{\emptyset}(X) \in \mathbf{T}_{\mathbf{1}}(X)$ and $\mathbf{T}_{\emptyset}(X) \neq \mathbf{T}_{\mathbf{1}}(X)$.

(27) If X is transitive, then $\mathbf{T}(X)$ is transitive.

(28) If $Y \in \mathbf{T}(X)$, then $\overline{\overline{Y}} < \overline{\overline{\mathbf{T}(X)}}$.

(29) If $Y \in \mathbf{T}(X)$, then $Y \not\approx \mathbf{T}(X)$.

(30) If $x \in \mathbf{T}(X)$ and $y \in \mathbf{T}(X)$, then $\{x\} \in \mathbf{T}(X)$ and $\{x,y\} \in \mathbf{T}(X)$.

(31) If $x \in \mathbf{T}(X)$ and $y \in \mathbf{T}(X)$, then $\langle x, y\rangle \in \mathbf{T}(X)$.

(32) If $Y \subseteq \mathbf{T}(X)$ and $Z \subseteq \mathbf{T}(X)$, then $[:Y, Z:] \subseteq \mathbf{T}(X)$.

Let us consider A. The functor $\mathbf{R}_A$ is defined by the condition (Def. 6).

(Def. 6) There exists L such that

(i) $\mathbf{R}_A = \operatorname{last} L$,

(ii) $\operatorname{dom} L = \operatorname{succ} A$,

(iii) $L(\emptyset) = \emptyset$,

(iv) for every C such that $\operatorname{succ} C \in \operatorname{succ} A$ holds $L(\operatorname{succ} C) = 2^{L(C)}$, and

(v) for every C such that $C \in \operatorname{succ} A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L(C) = \bigcup \operatorname{rng}(L \restriction C)$.

Next we state a number of propositions:

(33) $\mathbf{R}_\emptyset = \emptyset$.

(34) $\mathbf{R}_{\operatorname{succ} A} = 2^{\mathbf{R}_A}$.

(35) If $A \neq \emptyset$ and A is a limit ordinal number, then for every x holds $x \in \mathbf{R}_A$ iff there exists B such that $B \in A$ and $x \in \mathbf{R}_B$.

(36) $X \subseteq \mathbf{R}_A$ iff $X \in \mathbf{R}_{\operatorname{succ} A}$.

(37) $\mathbf{R}_A$ is transitive.

(38) If $X \in \mathbf{R}_A$, then $X \subseteq \mathbf{R}_A$.

(39) $\mathbf{R}_A \subseteq \mathbf{R}_{\operatorname{succ} A}$.

(40) $\bigcup(\mathbf{R}_A) \subseteq \mathbf{R}_A$.

(41) If $X \in \mathbf{R}_A$, then $\bigcup X \in \mathbf{R}_A$.

(42) $A \in B$ iff $\mathbf{R}_A \in \mathbf{R}_B$.

(43) $A \subseteq B$ iff $\mathbf{R}_A \subseteq \mathbf{R}_B$.

(44) $A \subseteq \mathbf{R}_A$.

(45) For all A, X such that $X \in \mathbf{R}_A$ holds $X \not\approx \mathbf{R}_A$ and $\overline{\overline{X}} < \overline{\overline{\mathbf{R}_A}}$.

(46) $X \subseteq \mathbf{R}_A$ iff $2^X \subseteq \mathbf{R}_{\operatorname{succ} A}$.

(47) If $X \subseteq Y$ and $Y \in \mathbf{R}_A$, then $X \in \mathbf{R}_A$.

(48) $X \in \mathbf{R}_A$ iff $2^X \in \mathbf{R}_{\operatorname{succ} A}$.

(49) $x \in \mathbf{R}_A$ iff $\{x\} \in \mathbf{R}_{\operatorname{succ} A}$.

(50) $x \in \mathbf{R}_A$ and $y \in \mathbf{R}_A$ iff $\{x,y\} \in \mathbf{R}_{\operatorname{succ} A}$.

(51) $x \in \mathbf{R}_A$ and $y \in \mathbf{R}_A$ iff $\langle x, y\rangle \in \mathbf{R}_{\operatorname{succ}\operatorname{succ} A}$.

(52) If X is transitive and $\mathbf{R}_A \cap \mathbf{T}(X) = \mathbf{R}_{\operatorname{succ} A} \cap \mathbf{T}(X)$, then $\mathbf{T}(X) \subseteq \mathbf{R}_A$.

(53) If X is transitive, then there exists A such that $\mathbf{T}(X) \subseteq \mathbf{R}_A$.

(54) If X is transitive, then $\bigcup X \subseteq X$.

(55) If X is transitive and Y is transitive, then $X \cup Y$ is transitive.

(56) If X is transitive and Y is transitive, then $X \cap Y$ is transitive.

In the sequel k, n denote natural numbers.
Let us consider X. The functor $X^{*\in}$ yielding a set is defined by:

(Def. 7) $x \in X^{*\in}$ iff there exist f, n such that $x \in f(n)$ and $\mathrm{dom}\, f = \mathbb{N}$ and $f(0) = X$ and for every k holds $f(k+1) = \bigcup f(k)$.

We now state a number of propositions:

(58)[3] $X^{*\in}$ is transitive.

(59) $X \subseteq X^{*\in}$.

(60) If $X \subseteq Y$ and Y is transitive, then $X^{*\in} \subseteq Y$.

(61) If for every Z such that $X \subseteq Z$ and Z is transitive holds $Y \subseteq Z$ and $X \subseteq Y$ and Y is transitive, then $X^{*\in} = Y$.

(62) If X is transitive, then $X^{*\in} = X$.

(63) $\emptyset^{*\in} = \emptyset$.

(64) $A^{*\in} = A$.

(65) If $X \subseteq Y$, then $X^{*\in} \subseteq Y^{*\in}$.

(66) $(X^{*\in})^{*\in} = X^{*\in}$.

(67) $(X \cup Y)^{*\in} = X^{*\in} \cup Y^{*\in}$.

(68) $(X \cap Y)^{*\in} \subseteq X^{*\in} \cap Y^{*\in}$.

(69) There exists A such that $X \subseteq \mathbf{R}_A$.

Let us consider X. The functor $\mathrm{rk}(X)$ yields an ordinal number and is defined as follows:

(Def. 8) $X \subseteq \mathbf{R}_{\mathrm{rk}(X)}$ and for every B such that $X \subseteq \mathbf{R}_B$ holds $\mathrm{rk}(X) \subseteq B$.

Next we state a number of propositions:

(71)[4] $\mathrm{rk}(2^X) = \mathrm{succ}\,\mathrm{rk}(X)$.

(72) $\mathrm{rk}(\mathbf{R}_A) = A$.

(73) $X \subseteq \mathbf{R}_A$ iff $\mathrm{rk}(X) \subseteq A$.

(74) $X \in \mathbf{R}_A$ iff $\mathrm{rk}(X) \in A$.

(75) If $X \subseteq Y$, then $\mathrm{rk}(X) \subseteq \mathrm{rk}(Y)$.

(76) If $X \in Y$, then $\mathrm{rk}(X) \in \mathrm{rk}(Y)$.

(77) $\mathrm{rk}(X) \subseteq A$ iff for every Y such that $Y \in X$ holds $\mathrm{rk}(Y) \in A$.

(78) $A \subseteq \mathrm{rk}(X)$ iff for every B such that $B \in A$ there exists Y such that $Y \in X$ and $B \subseteq \mathrm{rk}(Y)$.

(79) $\mathrm{rk}(X) = \emptyset$ iff $X = \emptyset$.

(80) If $\mathrm{rk}(X) = \mathrm{succ}\,A$, then there exists Y such that $Y \in X$ and $\mathrm{rk}(Y) = A$.

(81) $\mathrm{rk}(A) = A$.

(82) $\mathrm{rk}(\mathbf{T}(X)) \neq \emptyset$ and $\mathrm{rk}(\mathbf{T}(X))$ is a limit ordinal number.

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[3] The proposition (57) has been removed.
[4] The proposition (70) has been removed.

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