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# Introduction to Circuits, II[1]

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**Summary.** This article is the last in a series of four articles (preceded by [22], [23], [21]) about modelling circuits by many sorted algebras.

The notion of a circuit computation is defined as a sequence of circuit states. For a state of a circuit the next state is given by executing operations at circuit vertices in the current state, according to denotations of the operations. The values at input vertices at each state of a computation are provided by an external sequence of input values. The process of how input values propagate through a circuit is described in terms of a homomorphism of the free envelope algebra of the circuit into itself. We prove that every computation of a circuit over a finite monotonic signature and with constant input values stabilizes after executing the number of steps equal to the depth of the circuit.

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The articles [25], [13], [32], [29], [33], [28], [11], [12], [18], [14], [2], [8], [16], [5], [6], [7], [30], [1], [3], [31], [4], [15], [9], [26], [19], [27], [10], [20], [17], [24], [22], [23], and [21] provide the notation and terminology for this paper.

## 1. Circuit Inputs

In this paper $I_1$ denotes a monotonic circuit-like non void non empty many sorted signature.

The following proposition is true

(1) Let $X$ be a non-empty many sorted set indexed by the carrier of $I_1$, $H$ be a many sorted function from $\mathrm{Free}(X)$ into $\mathrm{Free}(X)$, $H_1$ be a function yielding function, $v$ be a sort symbol of $I_1$, $p$ be a decorated tree yielding finite sequence, and $t$ be an element of (the sorts of $\mathrm{Free}(X))(v)$. Suppose that

- (i) $v \in \mathrm{InnerVertices}(I_1)$,
- (ii) $t = \langle$the action at $v$, the carrier of $I_1\rangle$-tree$(p)$,
- (iii) $H$ is a homomorphism of $\mathrm{Free}(X)$ into $\mathrm{Free}(X)$, and
- (iv) $H_1 = H \cdot \mathrm{Arity}($the action at $v)$.

Then there exists a decorated tree yielding finite sequence $H_2$ such that $H_2 = H_1 \hookleftarrow p$ and $H(v)(t) = \langle$the action at $v$, the carrier of $I_1\rangle$-tree$(H_2)$.

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Let us consider $I_1$, let $S_1$ be a non-empty circuit of $I_1$, let $s$ be a state of $S_1$, and let $i_1$ be an input assignment of $S_1$. Then $s+\cdot i_1$ is a state of $S_1$.

Let us consider $I_1$, let $A$ be a non-empty circuit of $I_1$, and let $i_1$ be an input assignment of $A$. The functor FixInput($i_1$) yielding a many sorted function from FreeGenerator(the sorts of $A$) into the sorts of FreeEnvelope($A$) is defined by the condition (Def. 1).

(Def. 1) Let $v$ be a vertex of $I_1$. Then

(i) if $v \in \mathrm{InputVertices}(I_1)$, then $(\mathrm{FixInput}(i_1))(v) = \mathrm{FreeGenerator}(v, \text{the sorts of } A) \longmapsto$ the root tree of $\langle i_1(v), v\rangle$,

(ii) if $v \in \mathrm{SortsWithConstants}(I_1)$, then $(\mathrm{FixInput}(i_1))(v) = \mathrm{FreeGenerator}(v, \text{the sorts of } A) \longmapsto$ the root tree of $\langle$the action at $v$, the carrier of $I_1\rangle$, and

(iii) if $v \in \mathrm{InnerVertices}(I_1) \setminus \mathrm{SortsWithConstants}(I_1)$, then $(\mathrm{FixInput}(i_1))(v) = \mathrm{id}_{\mathrm{FreeGenerator}(v, \text{the sorts of } A)}$.

Let us consider $I_1$, let $A$ be a non-empty circuit of $I_1$, and let $i_1$ be an input assignment of $A$. The functor FixInputExt($i_1$) yielding a many sorted function from FreeEnvelope($A$) into FreeEnvelope($A$) is defined as follows:

(Def. 2) FixInputExt($i_1$) is a homomorphism of FreeEnvelope($A$) into FreeEnvelope($A$) and $\mathrm{FixInput}(i_1) \subseteq \mathrm{FixInputExt}(i_1)$.

We now state several propositions:

(2) Let $A$ be a non-empty circuit of $I_1$, $i_1$ be an input assignment of $A$, $v$ be a vertex of $I_1$, $e$ be an element of (the sorts of FreeEnvelope($A$))($v$), and $x$ be a set. If $v \in \mathrm{InnerVertices}(I_1) \setminus \mathrm{SortsWithConstants}(I_1)$ and $e =$ the root tree of $\langle x, v\rangle$, then $(\mathrm{FixInputExt}(i_1))(v)(e) = e$.

(3) Let $A$ be a non-empty circuit of $I_1$, $i_1$ be an input assignment of $A$, $v$ be a vertex of $I_1$, and $x$ be an element of (the sorts of $A$)($v$). If $v \in \mathrm{InputVertices}(I_1)$, then $(\mathrm{FixInputExt}(i_1))(v)$(the root tree of $\langle x, v\rangle$) = the root tree of $\langle i_1(v), v\rangle$.

(4) Let $A$ be a non-empty circuit of $I_1$, $i_1$ be an input assignment of $A$, $v$ be a vertex of $I_1$, $e$ be an element of (the sorts of FreeEnvelope($A$))($v$), and $p$, $q$ be decorated tree yielding finite sequences. Suppose that

(i) $v \in \mathrm{InnerVertices}(I_1)$,

(ii) $e = \langle$the action at $v$, the carrier of $I_1\rangle$-tree($p$),

(iii) $\mathrm{dom}\, p = \mathrm{dom}\, q$, and

(iv) for every natural number $k$ such that $k \in \mathrm{dom}\, p$ holds $q(k) = (\mathrm{FixInputExt}(i_1))(\mathrm{Arity}(\text{the action at } v)_k)(p(k))$.

Then $(\mathrm{FixInputExt}(i_1))(v)(e) = \langle$the action at $v$, the carrier of $I_1\rangle$-tree($q$).

(5) Let $A$ be a non-empty circuit of $I_1$, $i_1$ be an input assignment of $A$, $v$ be a vertex of $I_1$, and $e$ be an element of (the sorts of FreeEnvelope($A$))($v$). Suppose $v \in \mathrm{SortsWithConstants}(I_1)$. Then $(\mathrm{FixInputExt}(i_1))(v)(e) =$ the root tree of $\langle$the action at $v$, the carrier of $I_1\rangle$.

(6) Let $A$ be a non-empty circuit of $I_1$, $i_1$ be an input assignment of $A$, $v$ be a vertex of $I_1$, $e$, $e_1$ be elements of (the sorts of FreeEnvelope($A$))($v$), and $t$, $t_1$ be decorated trees. If $t = e$ and $t_1 = e_1$ and $e_1 = (\mathrm{FixInputExt}(i_1))(v)(e)$, then $\mathrm{dom}\, t = \mathrm{dom}\, t_1$.

(7) Let $A$ be a non-empty circuit of $I_1$, $i_1$ be an input assignment of $A$, $v$ be a vertex of $I_1$, and $e$, $e_1$ be elements of (the sorts of FreeEnvelope($A$))($v$). If $e_1 = (\mathrm{FixInputExt}(i_1))(v)(e)$, then $\mathrm{card}\, e = \mathrm{card}\, e_1$.

Let us consider $I_1$, let $S_1$ be a non-empty circuit of $I_1$, let $v$ be a vertex of $I_1$, and let $i_1$ be an input assignment of $S_1$. The functor InputGenTree($v, i_1$) yielding an element of (the sorts of FreeEnvelope($S_1$))($v$) is defined as follows:

(Def. 3) There exists an element $e$ of (the sorts of FreeEnvelope($S_1$))($v$) such that $\mathrm{card}\, e = \mathrm{size}(v, S_1)$ and $\mathrm{InputGenTree}(v, i_1) = (\mathrm{FixInputExt}(i_1))(v)(e)$.

We now state two propositions:

(8) Let $S_1$ be a non-empty circuit of $I_1$, $v$ be a vertex of $I_1$, and $i_1$ be an input assignment of $S_1$. Then $\text{InputGenTree}(v,i_1) = (\text{FixInputExt}(i_1))(v)(\text{InputGenTree}(v,i_1))$.

(9) Let $S_1$ be a non-empty circuit of $I_1$, $v$ be a vertex of $I_1$, $i_1$ be an input assignment of $S_1$, and $p$ be a decorated tree yielding finite sequence. Suppose that

(i) $v \in \text{InnerVertices}(I_1)$,

(ii) $\text{dom}\, p = \text{dom}\, \text{Arity}(\text{the action at } v)$, and

(iii) for every natural number $k$ such that $k \in \text{dom}\, p$ holds $p(k) = \text{InputGenTree}(\text{Arity}(\text{the action at } v)_k, i_1)$.

Then $\text{InputGenTree}(v,i_1) = \langle \text{the action at } v, \text{ the carrier of } I_1 \rangle\text{-tree}(p)$.

Let us consider $I_1$, let $S_1$ be a non-empty circuit of $I_1$, let $v$ be a vertex of $I_1$, and let $i_1$ be an input assignment of $S_1$. The functor $\text{InputGenValue}(v,i_1)$ yielding an element of (the sorts of $S_1)(v)$ is defined as follows:

(Def. 4) $\text{InputGenValue}(v,i_1) = (\text{Eval}(S_1))(v)(\text{InputGenTree}(v,i_1))$.

One can prove the following propositions:

(10) Let $S_1$ be a non-empty circuit of $I_1$, $v$ be a vertex of $I_1$, and $i_1$ be an input assignment of $S_1$. If $v \in \text{InputVertices}(I_1)$, then $\text{InputGenValue}(v,i_1) = i_1(v)$.

(11) Let $S_1$ be a non-empty circuit of $I_1$, $v$ be a vertex of $I_1$, and $i_1$ be an input assignment of $S_1$. If $v \in \text{SortsWithConstants}(I_1)$, then $\text{InputGenValue}(v,i_1) = (\text{Set-Constants}(S_1))(v)$.

## 2. Circuit Computations

Let $I_1$ be a circuit-like non void non empty many sorted signature, let $S_1$ be a non-empty circuit of $I_1$, and let $s$ be a state of $S_1$. The functor $\text{Following}(s)$ yields a state of $S_1$ and is defined by the condition (Def. 5).

(Def. 5) Let $v$ be a vertex of $I_1$. Then

(i) if $v \in \text{InputVertices}(I_1)$, then $(\text{Following}(s))(v) = s(v)$, and

(ii) if $v \in \text{InnerVertices}(I_1)$, then $(\text{Following}(s))(v) = (\text{Den}(\text{the action at } v, S_1))((\text{the action at } v) \text{ depends-on-in}\, s)$.

The following proposition is true

(12) Let $S_1$ be a non-empty circuit of $I_1$, $s$ be a state of $S_1$, and $i_1$ be an input assignment of $S_1$. If $i_1 \subseteq s$, then $i_1 \subseteq \text{Following}(s)$.

Let $I_1$ be a circuit-like non void non empty many sorted signature, let $S_1$ be a non-empty circuit of $I_1$, and let $I_2$ be a state of $S_1$. We say that $I_2$ is stable if and only if:

(Def. 6) $I_2 = \text{Following}(I_2)$.

Let us consider $I_1$, let $S_1$ be a non-empty circuit of $I_1$, let $s$ be a state of $S_1$, and let $i_1$ be an input assignment of $S_1$. The functor $\text{Following}(s,i_1)$ yields a state of $S_1$ and is defined by:

(Def. 7) $\text{Following}(s,i_1) = \text{Following}(s {+}\!\cdot i_1)$.

Let us consider $I_1$, let $S_1$ be a non-empty circuit of $I_1$, let $I_3$ be an input function of $S_1$, and let $s$ be a state of $S_1$. The functor $\text{InitialComp}(s,I_3)$ yielding a state of $S_1$ is defined as follows:

(Def. 8) $\text{InitialComp}(s,I_3) = s {+}\!\cdot (0\text{-}th\text{-input}(I_3)) {+}\!\cdot \text{Set-Constants}(S_1)$.

Let us consider $I_1$, let $S_1$ be a non-empty circuit of $I_1$, let $I_3$ be an input function of $S_1$, and let $s$ be a state of $S_1$. The functor Computation$(s,I_3)$ yields a function from $\mathbb{N}$ into $\prod$(the sorts of $S_1$) and is defined by:

(Def. 9) $(\text{Computation}(s,I_3))(0) = \text{InitialComp}(s,I_3)$ and for every natural number $i$ holds $(\text{Computation}(s,I_3))(i+1) = \text{Following}((\text{Computation}(s,I_3))(i), (i+1)\text{-}th\text{-input}(I_3))$.

In the sequel $S_1$ is a non-empty circuit of $I_1$, $s$ is a state of $S_1$, and $i_1$ is an input assignment of $S_1$.

We now state the proposition

(13) Let $k$ be a natural number. Suppose that for every vertex $v$ of $I_1$ such that $\text{depth}(v,S_1) \leq k$ holds $s(v) = \text{InputGenValue}(v,i_1)$. Let $v_1$ be a vertex of $I_1$. If $\text{depth}(v_1,S_1) \leq k+1$, then $(\text{Following}(s))(v_1) = \text{InputGenValue}(v_1,i_1)$.

For simplicity, we adopt the following rules: $I_1$ is a finite monotonic circuit-like non void non empty many sorted signature, $S_1$ is a non-empty circuit of $I_1$, $I_3$ is an input function of $S_1$, $s$ is a state of $S_1$, and $i_1$ is an input assignment of $S_1$.

The following propositions are true:

(14) If commute$(I_3)$ is constant and InputVertices$(I_1)$ is non empty, then for all $s$, $i_1$ such that $i_1 = (\text{commute}(I_3))(0)$ and for every natural number $k$ holds $i_1 \subseteq (\text{Computation}(s,I_3))(k)$.

(15) Let $n$ be a natural number. Suppose commute$(I_3)$ is constant and InputVertices$(I_1)$ is non empty and $(\text{Computation}(s,I_3))(n)$ is stable. Let $m$ be a natural number. If $n \leq m$, then $(\text{Computation}(s,I_3))(n) = (\text{Computation}(s,I_3))(m)$.

(16) Suppose commute$(I_3)$ is constant and InputVertices$(I_1)$ is non empty. Let given $s$, $i_1$. Suppose $i_1 = (\text{commute}(I_3))(0)$. Let $k$ be a natural number and $v$ be a vertex of $I_1$. If $\text{depth}(v,S_1) \leq k$, then $((\text{Computation}(s,I_3))(k)$ **qua** element of $\prod(\text{the sorts of } S_1))(v) = \text{InputGenValue}(v,i_1)$.

(17) Suppose commute$(I_3)$ is constant and InputVertices$(I_1)$ is non empty and $i_1 = (\text{commute}(I_3))(0)$. Let $s$ be a state of $S_1$, $v$ be a vertex of $I_1$, and $n$ be an element of $\mathbb{N}$. If $n = \text{depth}(S_1)$, then $((\text{Computation}(s,I_3))(n)$ **qua** state of $S_1)(v) = \text{InputGenValue}(v,i_1)$.

(18) Suppose commute$(I_3)$ is constant and InputVertices$(I_1)$ is non empty. Let $s$ be a state of $S_1$ and $n$ be an element of $\mathbb{N}$. If $n = \text{depth}(S_1)$, then $(\text{Computation}(s,I_3))(n)$ is stable.

(19) If commute$(I_3)$ is constant and InputVertices$(I_1)$ is non empty, then for all states $s_1$, $s_2$ of $S_1$ holds $(\text{Computation}(s_1,I_3))(\text{depth}(S_1)) = (\text{Computation}(s_2,I_3))(\text{depth}(S_1))$.

## REFERENCES


[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/card_1.html.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[3] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/card_3.html.

[4] Grzegorz Bancerek. Cartesian product of functions. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/funct_6.html.

[5] Grzegorz Bancerek. König's Lemma. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/trees_2.html.

[6] Grzegorz Bancerek. Sets and functions of trees and joining operations of trees. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/trees_3.html.

[7] Grzegorz Bancerek. Joining of decorated trees. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/trees_4.html.

[8] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[9] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/dtconstr.html.

[10] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_2.html.

[11] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[12] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[13] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[14] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/funct_4.html.

[15] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar — part I. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/lang1.html.

[16] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[17] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_3.html.

[18] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seqm_3.html.

[19] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pralg_1.html.

[20] Beata Madras. Products of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/pralg_2.html.

[21] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Introduction to circuits, I. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/circuit1.html.

[22] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/pre_circ.html.

[23] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, II. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msafree2.html.

[24] Beata Perkowska. Free many sorted universal algebra. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msafree.html.

[25] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[26] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[27] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_1.html.

[28] Andrzej Trybulec. On the sets inhabited by numbers. *Journal of Formalized Mathematics*, 15, 2003. http://mizar.org/JFM/Vol16/membered.html.

[29] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[30] Wojciech A. Trybulec. Groups. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/group_1.html.

[31] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[32] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[33] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

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