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Algebra of Morphisms

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The articles [17], [10], [23], [21], [18], [24], [7], [8], [6], [11], [15], [2], [22], [1], [3], [19], [14], [20], [13], [16], [12], [4], [5], and [9] provide the notation and terminology for this paper.

1. Preliminaries

Let I be a set and let A, f be functions. The functor $f\restriction_I A$ yields a many sorted function indexed by I and is defined as follows:

(Def. 1) For every set i such that $i \in I$ holds $(f\restriction_I A)(i) = f\restriction A(i)$.

One can prove the following propositions:

(1) For every set I and for every many sorted set A indexed by I holds $\mathrm{id}_{\bigcup A}\restriction_I A = \mathrm{id}_A$.

(2) Let I be a set, A, B be many sorted sets indexed by I, and f, g be functions. If $\mathrm{rng}_\kappa(f\restriction_I A)(\kappa) \subseteq B$, then $(g \cdot f)\restriction_I A = (g\restriction_I B) \circ (f\restriction_I A)$.

(3) Let f be a function, I be a set, and A, B be many sorted sets indexed by I. Suppose that for every set i such that $i \in I$ holds $A(i) \subseteq \mathrm{dom}\, f$ and $f^\circ A(i) \subseteq B(i)$. Then $f\restriction_I A$ is a many sorted function from A into B.

(4) Let A be a set, i be a natural number, and p be a finite sequence. Then $p \in A^i$ if and only if $\mathrm{len}\, p = i$ and $\mathrm{rng}\, p \subseteq A$.

(5) Let A be a set, i be a natural number, and p be a finite sequence of elements of A. Then $p \in A^i$ if and only if $\mathrm{len}\, p = i$.

(6) For every set A and for every natural number i holds $A^i \subseteq A^*$.

(7) For every set A and for every natural number i holds $i \neq 0$ and $A = \emptyset$ iff $A^i = \emptyset$.

(8) For all sets A, x holds $x \in A^1$ iff there exists a set a such that $a \in A$ and $x = \langle a \rangle$.

(9) For all sets A, a such that $\langle a \rangle \in A^1$ holds $a \in A$.

(10) For all sets A, x holds $x \in A^2$ iff there exist sets a, b such that $a \in A$ and $b \in A$ and $x = \langle a, b \rangle$.

(11) For all sets A, a, b such that $\langle a, b \rangle \in A^2$ holds $a \in A$ and $b \in A$.

(12) For all sets A, x holds $x \in A^3$ iff there exist sets a, b, c such that $a \in A$ and $b \in A$ and $c \in A$ and $x = \langle a,b,c \rangle$.

(13) For all sets A, a, b, c such that $\langle a,b,c \rangle \in A^3$ holds $a \in A$ and $b \in A$ and $c \in A$.

Let S be a non empty many sorted signature and let A be an algebra over S. We say that A is empty if and only if:

(Def. 3)[1] The sorts of A are empty yielding.

Let S be a non empty many sorted signature. Note that every algebra over S which is non-empty is also non empty.

Let S be a non empty non void many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Observe that $\mathrm{Free}(X)$ is disjoint valued.

Let S be a non empty non void many sorted signature. One can check that there exists an algebra over S which is strict, non-empty, and disjoint valued.

Let S be a non empty non void many sorted signature and let A be a non empty algebra over S. Observe that the sorts of A is non empty yielding.

Let us observe that there exists a function which is non empty yielding.

2. Signature of a Category

Let A be a set. The functor $\mathrm{CatSign}(A)$ yielding a strict many sorted signature is defined by the conditions (Def. 5).

(Def. 5)[2] (i) The carrier of $\mathrm{CatSign}(A) = [: \{0\}, A^2 :]$,

(ii) the operation symbols of $\mathrm{CatSign}(A) = [: \{1\}, A^1 :] \cup [: \{2\}, A^3 :]$,

(iii) for every set a such that $a \in A$ holds (the arity of $\mathrm{CatSign}(A))(\langle 1, \langle a \rangle \rangle) = \emptyset$ and (the result sort of $\mathrm{CatSign}(A))(\langle 1, \langle a \rangle \rangle) = \langle 0, \langle a,a \rangle \rangle$, and

(iv) for all sets a, b, c such that $a \in A$ and $b \in A$ and $c \in A$ holds (the arity of $\mathrm{CatSign}(A))(\langle 2, \langle a,b,c \rangle \rangle) = \langle \langle 0, \langle b,c \rangle \rangle, \langle 0, \langle a,b \rangle \rangle \rangle$ and (the result sort of $\mathrm{CatSign}(A))(\langle 2, \langle a,b,c \rangle \rangle) = \langle 0, \langle a,c \rangle \rangle$.

Let A be a set. One can verify that $\mathrm{CatSign}(A)$ is feasible.

Let A be a non empty set. One can verify that $\mathrm{CatSign}(A)$ is non empty and non void.

Instead of a feasible many sorted signature we will use a signature.

Let S be a signature. We say that S is categorial if and only if:

(Def. 6) There exists a set A such that $\mathrm{CatSign}(A)$ is a subsignature of S and the carrier of $S = [: \{0\}, A^2 :]$.

One can verify that every non empty signature which is categorial is also non void.

One can verify that there exists a signature which is categorial, non empty, and strict.

A cat-signature is a categorial signature.

Let A be a set. A signature is called a cat-signature of A if:

(Def. 7) $\mathrm{CatSign}(A)$ is a subsignature of it and the carrier of it $= [: \{0\}, A^2 :]$.

One can prove the following proposition

(14) For all sets A_1, A_2 and for every cat-signature S of A_1 such that S is a cat-signature of A_2 holds $A_1 = A_2$.

Let A be a set. Note that every cat-signature of A is categorial.

Let A be a non empty set. Observe that every cat-signature of A is non empty.

Let A be a set. Observe that there exists a cat-signature of A which is strict.

Let A be a set. Then $\mathrm{CatSign}(A)$ is a strict cat-signature of A.

Let S be a many sorted signature. The functor underlay$S$ is defined by the condition (Def. 8).

[1] The definition (Def. 2) has been removed.

[2] The definition (Def. 4) has been removed.

(Def. 8) Let x be a set. Then $x \in \operatorname{underlay} S$ if and only if there exists a set a and there exists a function f such that $\langle a, f \rangle \in$ (the carrier of S) $\cup$ (the operation symbols of S) and $x \in \operatorname{rng} f$.

One can prove the following proposition

(15) For every set A holds $\operatorname{underlay} \operatorname{CatSign}(A) = A$.

Let S be a many sorted signature. We say that S is δ-concrete if and only if the condition (Def. 9) is satisfied.

(Def. 9) There exists a function f from $\mathbb{N}$ into $\mathbb{N}$ such that

(i) for every set s such that $s \in$ the carrier of S there exists a natural number i and there exists a finite sequence p such that $s = \langle i, p \rangle$ and $\operatorname{len} p = f(i)$ and $[: \{i\}, (\operatorname{underlay} S)^{f(i)} :] \subseteq$ the carrier of S, and

(ii) for every set o such that $o \in$ the operation symbols of S there exists a natural number i and there exists a finite sequence p such that $o = \langle i, p \rangle$ and $\operatorname{len} p = f(i)$ and $[: \{i\}, (\operatorname{underlay} S)^{f(i)} :] \subseteq$ the operation symbols of S.

Let A be a set. One can verify that $\operatorname{CatSign}(A)$ is δ-concrete.

One can check that there exists a cat-signature which is δ-concrete, non empty, and strict. Let A be a set. One can verify that there exists a cat-signature of A which is δ-concrete and strict.

We now state four propositions:

(16) Let S be a δ-concrete many sorted signature and x be a set. Suppose $x \in$ the carrier of S or $x \in$ the operation symbols of S. Then there exists a natural number i and there exists a finite sequence p such that $x = \langle i, p \rangle$ and $\operatorname{rng} p \subseteq \operatorname{underlay} S$.

(17) Let S be a δ-concrete many sorted signature, i be a set, and p_1, p_2 be finite sequences. Suppose that

(i) $\langle i, p_1 \rangle \in$ the carrier of S and $\langle i, p_2 \rangle \in$ the carrier of S, or

(ii) $\langle i, p_1 \rangle \in$ the operation symbols of S and $\langle i, p_2 \rangle \in$ the operation symbols of S.

Then $\operatorname{len} p_1 = \operatorname{len} p_2$.

(18) Let S be a δ-concrete many sorted signature, i be a set, and p_1, p_2 be finite sequences such that $\operatorname{len} p_2 = \operatorname{len} p_1$ and $\operatorname{rng} p_2 \subseteq \operatorname{underlay} S$. Then

(i) if $\langle i, p_1 \rangle \in$ the carrier of S, then $\langle i, p_2 \rangle \in$ the carrier of S, and

(ii) if $\langle i, p_1 \rangle \in$ the operation symbols of S, then $\langle i, p_2 \rangle \in$ the operation symbols of S.

(19) Every δ-concrete categorial non empty signature S is a cat-signature of $\operatorname{underlay} S$.

3. Symbols of Categorial Signatures

Let S be a non empty cat-signature and let s be a sort symbol of S. Observe that s_2 is relation-like and function-like.

Let S be a non empty δ-concrete many sorted signature and let s be a sort symbol of S. Observe that s_2 is relation-like and function-like.

Let S be a non void δ-concrete many sorted signature and let o be an element of the operation symbols of S. One can check that o_2 is relation-like and function-like.

Let S be a non empty cat-signature and let s be a sort symbol of S. Observe that s_2 is finite sequence-like.

Let S be a non empty δ-concrete many sorted signature and let s be a sort symbol of S. One can verify that s_2 is finite sequence-like.

Let S be a non void δ-concrete many sorted signature and let o be an element of the operation symbols of S. Observe that o_2 is finite sequence-like.

Let a be a set. The functor $\operatorname{idsym} a$ is defined by:

(Def. 10) $\operatorname{idsym} a = \langle 1, \langle a \rangle \rangle$.

Let b be a set. The functor homsym(a,b) is defined as follows:

(Def. 11) homsym$(a,b) = \langle 0, \langle a,b \rangle \rangle$.

Let c be a set. The functor compsym(a,b,c) is defined as follows:

(Def. 12) compsym$(a,b,c) = \langle 2, \langle a,b,c \rangle \rangle$.

The following proposition is true

(20) Let A be a non empty set, S be a cat-signature of A, and a be an element of A. Then
(i) idsym $a \in$ the operation symbols of S, and
(ii) for every element b of A holds homsym$(a,b) \in$ the carrier of S and for every element c of A holds compsym$(a,b,c) \in$ the operation symbols of S.

Let A be a non empty set and let a be an element of A. Then idsym a is an operation symbol of CatSign(A). Let b be an element of A. Then homsym(a,b) is a sort symbol of CatSign(A). Let c be an element of A. Then compsym(a,b,c) is an operation symbol of CatSign(A).

Next we state several propositions:

(21) For all sets a, b such that idsym $a =$ idsym b holds $a = b$.

(22) For all sets a_1, b_1, a_2, b_2 such that homsym$(a_1,a_2) =$ homsym(b_1,b_2) holds $a_1 = b_1$ and $a_2 = b_2$.

(23) For all sets a_1, b_1, a_2, b_2, a_3, b_3 such that compsym$(a_1,a_2,a_3) =$ compsym(b_1,b_2,b_3) holds $a_1 = b_1$ and $a_2 = b_2$ and $a_3 = b_3$.

(24) Let A be a non empty set, S be a cat-signature of A, and s be a sort symbol of S. Then there exist elements a, b of A such that $s =$ homsym(a,b).

(25) For every non empty set A and for every operation symbol o of CatSign(A) holds $o_{\mathbf{1}} = 1$ and len$(o_{\mathbf{2}}) = 1$ or $o_{\mathbf{1}} = 2$ and len$(o_{\mathbf{2}}) = 3$.

(26) Let A be a non empty set and o be an operation symbol of CatSign(A). If $o_{\mathbf{1}} = 1$ or len$(o_{\mathbf{2}}) = 1$, then there exists an element a of A such that $o =$ idsym a.

(27) Let A be a non empty set and o be an operation symbol of CatSign(A). If $o_{\mathbf{1}} = 2$ or len$(o_{\mathbf{2}}) = 3$, then there exist elements a, b, c of A such that $o =$ compsym(a,b,c).

(28) For every non empty set A and for every element a of A holds Arity(idsym $a) = \emptyset$ and the result sort of idsym $a =$ homsym(a,a).

(29) For every non empty set A and for all elements a, b, c of A holds Arity(compsym$(a,b,c)) = \langle$homsym(b,c), homsym$(a,b)\rangle$ and the result sort of compsym$(a,b,c) =$ homsym(a,c).

4. Signature homomorphism generated by a functor

Let C_1, C_2 be categories and let F be a functor from C_1 to C_2. The functor Υ_F yields a function from the carrier of CatSign(the objects of C_1) into the carrier of CatSign(the objects of C_2) and is defined by:

(Def. 13) For every sort symbol s of CatSign(the objects of C_1) holds $\Upsilon_F(s) = \langle 0, \mathrm{Obj}\, F \cdot s_{\mathbf{2}} \rangle$.

The functor Ψ_F yields a function from the operation symbols of CatSign(the objects of C_1) into the operation symbols of CatSign(the objects of C_2) and is defined as follows:

(Def. 14) For every operation symbol o of CatSign(the objects of C_1) holds $\Psi_F(o) = \langle o_{\mathbf{1}}, \mathrm{Obj}\, F \cdot o_{\mathbf{2}} \rangle$.

The following propositions are true:

(30) For all categories C_1, C_2 and for every functor F from C_1 to C_2 and for all objects a, b of C_1 holds $\Upsilon_F(\mathrm{homsym}(a,b)) = \mathrm{homsym}(F(a),F(b))$.

(31) For all categories C_1, C_2 and for every functor F from C_1 to C_2 and for every object a of C_1 holds $\Psi_F(\mathrm{idsym}\,a) = \mathrm{idsym}\,F(a)$.

(32) Let C_1, C_2 be categories, F be a functor from C_1 to C_2, and a, b, c be objects of C_1. Then $\Psi_F(\mathrm{compsym}(a,b,c)) = \mathrm{compsym}(F(a),F(b),F(c))$.

(33) Let C_1, C_2 be categories and F be a functor from C_1 to C_2. Then Υ_F and Ψ_F form morphism between CatSign(the objects of C_1) and CatSign(the objects of C_2).

5. Algebra of Morphisms

One can prove the following proposition

(34) For every non empty set C and for every algebra A over CatSign(C) and for every element a of C holds $\mathrm{Args}(\mathrm{idsym}\,a,A) = \{\emptyset\}$.

The scheme *CatAlgEx* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, a 5-ary functor $\mathcal{G}$ yielding a set, and a unary functor $\mathcal{H}$ yielding a set, and states that:

There exists a strict algebra A over CatSign($\mathcal{A}$) such that

(i) for all elements a, b of $\mathcal{A}$ holds (the sorts of A)$(\mathrm{homsym}(a,b)) = \mathcal{F}(a,b)$,

(ii) for every element a of $\mathcal{A}$ holds $(\mathrm{Den}(\mathrm{idsym}\,a,A))(\emptyset) = \mathcal{H}(a)$, and

(iii) for all elements a, b, c of $\mathcal{A}$ and for all elements f, g of $\mathcal{B}$ such that $f \in \mathcal{F}(a,b)$ and $g \in \mathcal{F}(b,c)$ holds $(\mathrm{Den}(\mathrm{compsym}(a,b,c),A))(\langle g, f\rangle) = \mathcal{G}(a,b,c,g,f)$

provided the parameters satisfy the following conditions:

- For all elements a, b of $\mathcal{A}$ holds $\mathcal{F}(a,b) \subseteq \mathcal{B}$,
- For every element a of $\mathcal{A}$ holds $\mathcal{H}(a) \in \mathcal{F}(a,a)$, and
- For all elements a, b, c of $\mathcal{A}$ and for all elements f, g of $\mathcal{B}$ such that $f \in \mathcal{F}(a,b)$ and $g \in \mathcal{F}(b,c)$ holds $\mathcal{G}(a,b,c,g,f) \in \mathcal{F}(a,c)$.

Let C be a category. The functor MSAlg(C) yields a strict algebra over CatSign(the objects of C) and is defined by the conditions (Def. 15).

(Def. 15)(i) For all objects a, b of C holds (the sorts of MSAlg(C))$(\mathrm{homsym}(a,b)) = \mathrm{hom}(a,b)$,

(ii) for every object a of C holds $(\mathrm{Den}(\mathrm{idsym}\,a,\mathrm{MSAlg}(C)))(\emptyset) = \mathrm{id}_a$, and

(iii) for all objects a, b, c of C and for all morphisms f, g of C such that $\mathrm{dom}\,f = a$ and $\mathrm{cod}\,f = b$ and $\mathrm{dom}\,g = b$ and $\mathrm{cod}\,g = c$ holds $(\mathrm{Den}(\mathrm{compsym}(a,b,c),\mathrm{MSAlg}(C)))(\langle g, f\rangle) = g \cdot f$.

Next we state two propositions:

(36)[3] For every category A and for every object a of A holds $\mathrm{Result}(\mathrm{idsym}\,a,\mathrm{MSAlg}(A)) = \mathrm{hom}(a,a)$.

(37) For every category A and for all objects a, b, c of A holds $\mathrm{Args}(\mathrm{compsym}(a,b,c),\mathrm{MSAlg}(A)) = \prod\langle\mathrm{hom}(b,c),\mathrm{hom}(a,b)\rangle$ and $\mathrm{Result}(\mathrm{compsym}(a,b,c),\mathrm{MSAlg}(A)) = \mathrm{hom}(a,c)$.

Let C be a category. Observe that MSAlg(C) is disjoint valued and feasible.

The following propositions are true:

(38) Let C_1, C_2 be categories and F be a functor from C_1 to C_2. Then $F\restriction_{\text{the carrier of CatSign(the objects of }C_1)}$the sorts of MSAlg($C_1$) is a many sorted function from MSAlg(C_1) into $\mathrm{MSAlg}(C_2)\restriction_{(\Upsilon_F,\Psi_F)}$ CatSign(the objects of C_1).

(39) Let C be a category, a, b, c be objects of C, and x be a set. Then $x \in \mathrm{Args}(\mathrm{compsym}(a,b,c),\mathrm{MSAlg}(C))$ if and only if there exist morphisms g, f of C such that $x = \langle g, f\rangle$ and $\mathrm{dom}\,f = a$ and $\mathrm{cod}\,f = b$ and $\mathrm{dom}\,g = b$ and $\mathrm{cod}\,g = c$.

[3] The proposition (35) has been removed.

(40) Let C_1, C_2 be categories, F be a functor from C_1 to C_2, a, b, c be objects of C_1, and f, g be morphisms of C_1. Suppose $f \in \hom(a,b)$ and $g \in \hom(b,c)$. Let x be an element of $\mathrm{Args}(\mathrm{compsym}(a,b,c),\mathrm{MSAlg}(C_1))$. Suppose $x = \langle g, f\rangle$. Let H be a many sorted function from $\mathrm{MSAlg}(C_1)$ into $\mathrm{MSAlg}(C_2)\restriction_{(\Upsilon_F,\Psi_F)}\mathrm{CatSign}(\text{the objects of } C_1)$. Suppose $H = F\restriction_{\text{the carrier of CatSign(the objects of } C_1)}\text{the sorts of } \mathrm{MSAlg}(C_1)$. Then $H\#x = \langle F(g), F(f)\rangle$.

(42)[4] Let C be a category, a, b, c be objects of C, and f, g be morphisms of C. If $f \in \hom(a,b)$ and $g \in \hom(b,c)$, then $(\mathrm{Den}(\mathrm{compsym}(a,b,c),\mathrm{MSAlg}(C)))(\langle g, f\rangle) = g \cdot f$.

(43) Let C be a category, a, b, c, d be objects of C, and f, g, h be morphisms of C. Suppose $f \in \hom(a,b)$ and $g \in \hom(b,c)$ and $h \in \hom(c,d)$. Then $(\mathrm{Den}(\mathrm{compsym}(a,c,d),$ $\mathrm{MSAlg}(C)))(\langle h, (\mathrm{Den}(\mathrm{compsym}(a,b,c),\mathrm{MSAlg}(C)))(\langle g, f\rangle)\rangle) = (\mathrm{Den}(\mathrm{compsym}(a,b,d),\mathrm{MSAlg}(C)))(\langle(\mathrm{Den}(\mathrm{comps}$ $\mathrm{MSAlg}(C)))(\langle h,g\rangle), f\rangle)$.

(44) Let C be a category, a, b be objects of C, and f be a morphism of C. If $f \in \hom(a,b)$, then $(\mathrm{Den}(\mathrm{compsym}(a,b,b),\mathrm{MSAlg}(C)))(\langle \mathrm{id}_b, f\rangle) = f$ and $(\mathrm{Den}(\mathrm{compsym}(a,a,b),\mathrm{MSAlg}(C)))(\langle f, \mathrm{id}_a\rangle) = f$.

(45) Let C_1, C_2 be categories and F be a functor from C_1 to C_2. Then there exists a many sorted function H from $\mathrm{MSAlg}(C_1)$ into $\mathrm{MSAlg}(C_2)\restriction_{(\Upsilon_F,\Psi_F)}\mathrm{CatSign}(\text{the objects of } C_1)$ such that

(i) $H = F\restriction_{\text{the carrier of CatSign(the objects of } C_1)}\text{the sorts of } \mathrm{MSAlg}(C_1)$, and

(ii) H is a homomorphism of $\mathrm{MSAlg}(C_1)$ into $\mathrm{MSAlg}(C_2)\restriction_{(\Upsilon_F,\Psi_F)}\mathrm{CatSign}(\text{the objects of } C_1)$.

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