

# Products and Coproducts in Categories

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**Summary.** A product and coproduct in categories are introduced. The concepts included correspond to that presented in [6].

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The articles [8], [10], [11], [1], [7], [4], [2], [9], [3], and [5] provide the notation and terminology for this paper.

## 1. INDEXED FAMILIES

For simplicity, we adopt the following rules:  $I$  is a set,  $x, x_1, x_2, y$  are sets,  $A$  is a non empty set,  $C, D$  are categories,  $a, b, c, d$  are objects of  $C$ , and  $f, g, h, k, p_1, p_2, q_1, q_2, i_1, i_2, j_1, j_2$  are morphisms of  $C$ .

Let us consider  $I, A$ , let  $F$  be a function from  $I$  into  $A$ , and let  $x$  be a set. Let us assume that  $x \in I$ . Then  $F_x$  can be characterized by the condition:

(Def. 1)  $F_x = F(x)$ .

The scheme *LambdaIdx* deals with a set  $\mathcal{A}$ , a non empty set  $\mathcal{B}$ , and a unary functor  $\mathcal{F}$  yielding an element of  $\mathcal{B}$ , and states that:

There exists a function  $F$  from  $\mathcal{A}$  into  $\mathcal{B}$  such that for every  $x$  such that  $x \in \mathcal{A}$  holds  $F_x = \mathcal{F}(x)$

for all values of the parameters.

We now state the proposition

(1) For all functions  $F_1, F_2$  from  $I$  into  $A$  such that for every  $x$  such that  $x \in I$  holds  $(F_1)_x = (F_2)_x$  holds  $F_1 = F_2$ .

The scheme *FuncIdx correctness* deals with a set  $\mathcal{A}$ , a non empty set  $\mathcal{B}$ , and a unary functor  $\mathcal{F}$  yielding an element of  $\mathcal{B}$ , and states that:

(i) There exists a function  $F$  from  $\mathcal{A}$  into  $\mathcal{B}$  such that for every  $x$  such that  $x \in \mathcal{A}$  holds  $F_x = \mathcal{F}(x)$ , and

(ii) for all functions  $F_1, F_2$  from  $\mathcal{A}$  into  $\mathcal{B}$  such that for every  $x$  such that  $x \in \mathcal{A}$  holds  $(F_1)_x = \mathcal{F}(x)$  and for every  $x$  such that  $x \in \mathcal{A}$  holds  $(F_2)_x = \mathcal{F}(x)$  holds  $F_1 = F_2$

for all values of the parameters.

Let us consider  $A, I$  and let  $a$  be an element of  $A$ . Then  $I \mapsto a$  is a function from  $I$  into  $A$ .

One can prove the following two propositions:

(2) For every element  $a$  of  $A$  such that  $x \in I$  holds  $(I \mapsto a)_x = a$ .

(7)<sup>1</sup> If  $x_1 \neq x_2$ , then for all elements  $y_1, y_2$  of  $A$  holds  $[x_1 \mapsto y_1, x_2 \mapsto y_2]_{x_1} = y_1$  and  $[x_1 \mapsto y_1, x_2 \mapsto y_2]_{x_2} = y_2$ .

## 2. INDEXED FAMILIES OF MORPHISMS

Let us consider  $C, I$  and let  $F$  be a function from  $I$  into the morphisms of  $C$ . The functor  $\text{dom}_\kappa F(\kappa)$  yields a function from  $I$  into the objects of  $C$  and is defined by:

(Def. 3)<sup>2</sup> For every  $x$  such that  $x \in I$  holds  $(\text{dom}_\kappa F(\kappa))_x = \text{dom}(F_x)$ .

The functor  $\text{cod}_\kappa F(\kappa)$  yielding a function from  $I$  into the objects of  $C$  is defined by:

(Def. 4) For every  $x$  such that  $x \in I$  holds  $(\text{cod}_\kappa F(\kappa))_x = \text{cod}(F_x)$ .

Next we state four propositions:

(8)  $\text{dom}_\kappa(I \mapsto f)(\kappa) = I \mapsto \text{dom } f$ .

(9)  $\text{cod}_\kappa(I \mapsto f)(\kappa) = I \mapsto \text{cod } f$ .

(10)  $\text{dom}_\kappa[x_1 \mapsto p_1, x_2 \mapsto p_2](\kappa) = [x_1 \mapsto \text{dom } p_1, x_2 \mapsto \text{dom } p_2]$ .

(11)  $\text{cod}_\kappa[x_1 \mapsto p_1, x_2 \mapsto p_2](\kappa) = [x_1 \mapsto \text{cod } p_1, x_2 \mapsto \text{cod } p_2]$ .

Let us consider  $C, I$  and let  $F$  be a function from  $I$  into the morphisms of  $C$ . The functor  $F^{\text{op}}$  yielding a function from  $I$  into the morphisms of  $C^{\text{op}}$  is defined by:

(Def. 5) For every  $x$  such that  $x \in I$  holds  $(F^{\text{op}})_x = (F_x)^{\text{op}}$ .

One can prove the following three propositions:

(12)  $(I \mapsto f)^{\text{op}} = I \mapsto f^{\text{op}}$ .

(13) If  $x_1 \neq x_2$ , then  $[x_1 \mapsto p_1, x_2 \mapsto p_2]^{\text{op}} = [x_1 \mapsto p_1^{\text{op}}, x_2 \mapsto p_2^{\text{op}}]$ .

(14) For every function  $F$  from  $I$  into the morphisms of  $C$  holds  $(F^{\text{op}})^{\text{op}} = F$ .

Let us consider  $C, I$  and let  $F$  be a function from  $I$  into the morphisms of  $C^{\text{op}}$ . The functor  ${}^{\text{op}}F$  yields a function from  $I$  into the morphisms of  $C$  and is defined by:

(Def. 6) For every  $x$  such that  $x \in I$  holds  $({}^{\text{op}}F)_x = {}^{\text{op}}(F_x)$ .

We now state three propositions:

(15) For every morphism  $f$  of  $C^{\text{op}}$  holds  ${}^{\text{op}}(I \mapsto f) = I \mapsto {}^{\text{op}}f$ .

(16) If  $x_1 \neq x_2$ , then for all morphisms  $p_1, p_2$  of  $C^{\text{op}}$  holds  ${}^{\text{op}}[x_1 \mapsto p_1, x_2 \mapsto p_2] = [x_1 \mapsto {}^{\text{op}}p_1, x_2 \mapsto {}^{\text{op}}p_2]$ .

(17) For every function  $F$  from  $I$  into the morphisms of  $C$  holds  ${}^{\text{op}}({}^{\text{op}}F) = F$ .

Let us consider  $C, I$ , let  $F$  be a function from  $I$  into the morphisms of  $C$ , and let us consider  $f$ . The functor  $F \cdot f$  yielding a function from  $I$  into the morphisms of  $C$  is defined by:

(Def. 7) For every  $x$  such that  $x \in I$  holds  $(F \cdot f)_x = F_x \cdot f$ .

The functor  $f \cdot F$  yielding a function from  $I$  into the morphisms of  $C$  is defined by:

(Def. 8) For every  $x$  such that  $x \in I$  holds  $(f \cdot F)_x = f \cdot F_x$ .

We now state four propositions:

<sup>1</sup> The propositions (3)–(6) have been removed.

<sup>2</sup> The definition (Def. 2) has been removed.

- (18) If  $x_1 \neq x_2$ , then  $[x_1 \mapsto p_1, x_2 \mapsto p_2] \cdot f = [x_1 \mapsto p_1 \cdot f, x_2 \mapsto p_2 \cdot f]$ .
- (19) If  $x_1 \neq x_2$ , then  $f \cdot [x_1 \mapsto p_1, x_2 \mapsto p_2] = [x_1 \mapsto f \cdot p_1, x_2 \mapsto f \cdot p_2]$ .
- (20) For every function  $F$  from  $I$  into the morphisms of  $C$  such that  $\text{dom}_\kappa F(\kappa) = I \mapsto \text{cod } f$  holds  $\text{dom}_\kappa(F \cdot f)(\kappa) = I \mapsto \text{dom } f$  and  $\text{cod}_\kappa(F \cdot f)(\kappa) = \text{cod}_\kappa F(\kappa)$ .
- (21) For every function  $F$  from  $I$  into the morphisms of  $C$  such that  $\text{cod}_\kappa F(\kappa) = I \mapsto \text{dom } f$  holds  $\text{dom}_\kappa(f \cdot F)(\kappa) = \text{dom}_\kappa F(\kappa)$  and  $\text{cod}_\kappa(f \cdot F)(\kappa) = I \mapsto \text{cod } f$ .

Let us consider  $C, I$  and let  $F, G$  be functions from  $I$  into the morphisms of  $C$ . The functor  $F \cdot G$  yielding a function from  $I$  into the morphisms of  $C$  is defined as follows:

(Def. 9) For every  $x$  such that  $x \in I$  holds  $(F \cdot G)_x = F_x \cdot G_x$ .

We now state four propositions:

- (22) For all functions  $F, G$  from  $I$  into the morphisms of  $C$  such that  $\text{dom}_\kappa F(\kappa) = \text{cod}_\kappa G(\kappa)$  holds  $\text{dom}_\kappa(F \cdot G)(\kappa) = \text{dom}_\kappa G(\kappa)$  and  $\text{cod}_\kappa(F \cdot G)(\kappa) = \text{cod}_\kappa F(\kappa)$ .
- (23) If  $x_1 \neq x_2$ , then  $[x_1 \mapsto p_1, x_2 \mapsto p_2] \cdot [x_1 \mapsto q_1, x_2 \mapsto q_2] = [x_1 \mapsto p_1 \cdot q_1, x_2 \mapsto p_2 \cdot q_2]$ .
- (24) For every function  $F$  from  $I$  into the morphisms of  $C$  holds  $F \cdot f = F \cdot (I \mapsto f)$ .
- (25) For every function  $F$  from  $I$  into the morphisms of  $C$  holds  $f \cdot F = (I \mapsto f) \cdot F$ .

### 3. RETRACTIONS AND CORETRACTIONS

Let us consider  $C$  and let  $I_1$  be a morphism of  $C$ . We say that  $I_1$  is retraction if and only if:

(Def. 10) There exists  $g$  such that  $\text{cod } g = \text{dom } I_1$  and  $I_1 \cdot g = \text{id}_{\text{cod } I_1}$ .

We say that  $I_1$  is coretraction if and only if:

(Def. 11) There exists  $g$  such that  $\text{dom } g = \text{cod } I_1$  and  $g \cdot I_1 = \text{id}_{\text{dom } I_1}$ .

We now state a number of propositions:

- (26) If  $f$  is retraction, then  $f$  is epi.
- (27) If  $f$  is coretraction, then  $f$  is monic.
- (28) If  $f$  is retraction and  $g$  is retraction and  $\text{dom } g = \text{cod } f$ , then  $g \cdot f$  is retraction.
- (29) If  $f$  is coretraction and  $g$  is coretraction and  $\text{dom } g = \text{cod } f$ , then  $g \cdot f$  is coretraction.
- (30) If  $\text{dom } g = \text{cod } f$  and  $g \cdot f$  is retraction, then  $g$  is retraction.
- (31) If  $\text{dom } g = \text{cod } f$  and  $g \cdot f$  is coretraction, then  $f$  is coretraction.
- (32) If  $f$  is retraction and monic, then  $f$  is invertible.
- (33) If  $f$  is coretraction and epi, then  $f$  is invertible.
- (34)  $f$  is invertible iff  $f$  is retraction and coretraction.
- (35) For every functor  $T$  from  $C$  to  $D$  such that  $f$  is retraction holds  $T(f)$  is retraction.
- (36) For every functor  $T$  from  $C$  to  $D$  such that  $f$  is coretraction holds  $T(f)$  is coretraction.
- (37)  $f$  is retraction iff  $f^{\text{op}}$  is coretraction.
- (38)  $f$  is coretraction iff  $f^{\text{op}}$  is retraction.

4. MORPHISMS DETERMINED BY A TERMINAL OBJECT

Let us consider  $C, a, b$ . Let us assume that  $b$  is terminal.

(Def. 12)  $|^a b$  is a morphism from  $a$  to  $b$ .

We now state three propositions:

- (39) If  $b$  is terminal, then  $\text{dom}|^a b = a$  and  $\text{cod}|^a b = b$ .
- (40) If  $b$  is terminal and  $\text{dom} f = a$  and  $\text{cod} f = b$ , then  $|^a b = f$ .
- (41) For every morphism  $f$  from  $a$  to  $b$  such that  $b$  is terminal holds  $|^a b = f$ .

5. MORPHISMS DETERMINED BY AN INITIAL OBJECT

Let us consider  $C, a, b$ . Let us assume that  $a$  is initial.

(Def. 13)  $\text{init}(a, b)$  is a morphism from  $a$  to  $b$ .

One can prove the following three propositions:

- (42) If  $a$  is initial, then  $\text{dom} \text{init}(a, b) = a$  and  $\text{cod} \text{init}(a, b) = b$ .
- (43) If  $a$  is initial and  $\text{dom} f = a$  and  $\text{cod} f = b$ , then  $\text{init}(a, b) = f$ .
- (44) For every morphism  $f$  from  $a$  to  $b$  such that  $a$  is initial holds  $\text{init}(a, b) = f$ .

6. PRODUCTS

Let us consider  $C, a, I$ . A function from  $I$  into the morphisms of  $C$  is said to be a projections family from  $a$  onto  $I$  if:

(Def. 14)  $\text{dom}_\kappa \text{it}(\kappa) = I \mapsto a$ .

One can prove the following propositions:

- (45) For every projections family  $F$  from  $a$  onto  $I$  such that  $x \in I$  holds  $\text{dom}(F_x) = a$ .
- (46) Every function from  $\emptyset$  into the morphisms of  $C$  is a projections family from  $a$  onto  $\emptyset$ .
- (47) If  $\text{dom} f = a$ , then  $\{y\} \mapsto f$  is a projections family from  $a$  onto  $\{y\}$ .
- (48) If  $\text{dom} p_1 = a$  and  $\text{dom} p_2 = a$ , then  $[x_1 \mapsto p_1, x_2 \mapsto p_2]$  is a projections family from  $a$  onto  $\{x_1, x_2\}$ .
- (50)<sup>3</sup> Let  $F$  be a projections family from  $a$  onto  $I$ . If  $\text{cod} f = a$ , then  $F \cdot f$  is a projections family from  $\text{dom} f$  onto  $I$ .
- (51) Let  $F$  be a function from  $I$  into the morphisms of  $C$  and  $G$  be a projections family from  $a$  onto  $I$ . If  $\text{dom}_\kappa F(\kappa) = \text{cod}_\kappa G(\kappa)$ , then  $F \cdot G$  is a projections family from  $a$  onto  $I$ .
- (52) For every projections family  $F$  from  $\text{cod} f$  onto  $I$  holds  $f^{\text{op}} \cdot F^{\text{op}} = (F \cdot f)^{\text{op}}$ .

Let us consider  $C, a, I$  and let  $F$  be a function from  $I$  into the morphisms of  $C$ . We say that  $a$  is a product w.r.t.  $F$  if and only if the conditions (Def. 15) are satisfied.

(Def. 15)(i)  $F$  is a projections family from  $a$  onto  $I$ , and

- (ii) for every  $b$  and for every projections family  $F'$  from  $b$  onto  $I$  such that  $\text{cod}_\kappa F(\kappa) = \text{cod}_\kappa F'(\kappa)$  there exists  $h$  such that  $h \in \text{hom}(b, a)$  and for every  $k$  such that  $k \in \text{hom}(b, a)$  holds for every  $x$  such that  $x \in I$  holds  $F_x \cdot k = F'_x$  iff  $h = k$ .

<sup>3</sup> The proposition (49) has been removed.

One can prove the following propositions:

- (53) Let  $F$  be a projections family from  $c$  onto  $I$  and  $F'$  be a projections family from  $d$  onto  $I$ . Suppose  $c$  is a product w.r.t.  $F$  and  $d$  is a product w.r.t.  $F'$  and  $\text{cod}_\kappa F(\kappa) = \text{cod}_\kappa F'(\kappa)$ . Then  $c$  and  $d$  are isomorphic.
- (54) Let  $F$  be a projections family from  $c$  onto  $I$ . Suppose  $c$  is a product w.r.t.  $F$  and for all  $x_1, x_2$  such that  $x_1 \in I$  and  $x_2 \in I$  holds  $\text{hom}(\text{cod}(F_{x_1}), \text{cod}(F_{x_2})) \neq \emptyset$ . Let given  $x$ . If  $x \in I$ , then  $F_x$  is retraction.
- (55) For every function  $F$  from  $\emptyset$  into the morphisms of  $C$  holds  $a$  is a product w.r.t.  $F$  iff  $a$  is terminal.
- (56) Let  $F$  be a projections family from  $a$  onto  $I$ . Suppose  $a$  is a product w.r.t.  $F$  and  $\text{dom } f = a$  and  $\text{cod } f = a$  and  $f$  is invertible. Then  $b$  is a product w.r.t.  $F \cdot f$ .
- (57)  $a$  is a product w.r.t.  $\{y\} \mapsto \text{id}_a$ .
- (58) Let  $F$  be a projections family from  $a$  onto  $I$ . Suppose  $a$  is a product w.r.t.  $F$  and for every  $x$  such that  $x \in I$  holds  $\text{cod}(F_x)$  is terminal. Then  $a$  is terminal.

Let us consider  $C, c, p_1, p_2$ . We say that  $c$  is a product w.r.t.  $p_1$  and  $p_2$  if and only if the conditions (Def. 16) are satisfied.

- (Def. 16)(i)  $\text{dom } p_1 = c$ ,
- (ii)  $\text{dom } p_2 = c$ , and
- (iii) for all  $d, f, g$  such that  $f \in \text{hom}(d, \text{cod } p_1)$  and  $g \in \text{hom}(d, \text{cod } p_2)$  there exists  $h$  such that  $h \in \text{hom}(d, c)$  and for every  $k$  such that  $k \in \text{hom}(d, c)$  holds  $p_1 \cdot k = f$  and  $p_2 \cdot k = g$  iff  $h = k$ .

Next we state several propositions:

- (59) If  $x_1 \neq x_2$ , then  $c$  is a product w.r.t.  $p_1$  and  $p_2$  iff  $c$  is a product w.r.t.  $[x_1 \mapsto p_1, x_2 \mapsto p_2]$ .
- (60) Suppose  $\text{hom}(c, a) \neq \emptyset$  and  $\text{hom}(c, b) \neq \emptyset$ . Let  $p_1$  be a morphism from  $c$  to  $a$  and  $p_2$  be a morphism from  $c$  to  $b$ . Then  $c$  is a product w.r.t.  $p_1$  and  $p_2$  if and only if for every  $d$  such that  $\text{hom}(d, a) \neq \emptyset$  and  $\text{hom}(d, b) \neq \emptyset$  holds  $\text{hom}(d, c) \neq \emptyset$  and for every morphism  $f$  from  $d$  to  $a$  and for every morphism  $g$  from  $d$  to  $b$  there exists a morphism  $h$  from  $d$  to  $c$  such that for every morphism  $k$  from  $d$  to  $c$  holds  $p_1 \cdot k = f$  and  $p_2 \cdot k = g$  iff  $h = k$ .
- (61) Suppose  $c$  is a product w.r.t.  $p_1$  and  $p_2$  and  $d$  is a product w.r.t.  $q_1$  and  $q_2$  and  $\text{cod } p_1 = \text{cod } q_1$  and  $\text{cod } p_2 = \text{cod } q_2$ . Then  $c$  and  $d$  are isomorphic.
- (62) If  $c$  is a product w.r.t.  $p_1$  and  $p_2$  and  $\text{hom}(\text{cod } p_1, \text{cod } p_2) \neq \emptyset$  and  $\text{hom}(\text{cod } p_2, \text{cod } p_1) \neq \emptyset$ , then  $p_1$  is retraction and  $p_2$  is retraction.
- (63) If  $c$  is a product w.r.t.  $p_1$  and  $p_2$  and  $h \in \text{hom}(c, c)$  and  $p_1 \cdot h = p_1$  and  $p_2 \cdot h = p_2$ , then  $h = \text{id}_c$ .
- (64) If  $c$  is a product w.r.t.  $p_1$  and  $p_2$  and  $\text{dom } f = d$  and  $\text{cod } f = c$  and  $f$  is invertible, then  $d$  is a product w.r.t.  $p_1 \cdot f$  and  $p_2 \cdot f$ .
- (65) If  $c$  is a product w.r.t.  $p_1$  and  $p_2$  and  $\text{cod } p_2$  is terminal, then  $c$  and  $\text{cod } p_1$  are isomorphic.
- (66) If  $c$  is a product w.r.t.  $p_1$  and  $p_2$  and  $\text{cod } p_1$  is terminal, then  $c$  and  $\text{cod } p_2$  are isomorphic.

## 7. COPRODUCTS

Let us consider  $C, c, I$ . A function from  $I$  into the morphisms of  $C$  is said to be an injections family into  $c$  on  $I$  if:

(Def. 17)  $\text{cod}_{\kappa}\text{it}(\kappa) = I \mapsto c$ .

One can prove the following propositions:

- (67) For every injections family  $F$  into  $c$  on  $I$  such that  $x \in I$  holds  $\text{cod}(F_x) = c$ .
- (68) Every function from  $\emptyset$  into the morphisms of  $C$  is an injections family into  $a$  on  $\emptyset$ .
- (69) If  $\text{cod } f = a$ , then  $\{y\} \mapsto f$  is an injections family into  $a$  on  $\{y\}$ .
- (70) If  $\text{cod } p_1 = c$  and  $\text{cod } p_2 = c$ , then  $[x_1 \mapsto p_1, x_2 \mapsto p_2]$  is an injections family into  $c$  on  $\{x_1, x_2\}$ .
- (72)<sup>4</sup> For every injections family  $F$  into  $b$  on  $I$  such that  $\text{dom } f = b$  holds  $f \cdot F$  is an injections family into  $\text{cod } f$  on  $I$ .
- (73) Let  $F$  be an injections family into  $b$  on  $I$  and  $G$  be a function from  $I$  into the morphisms of  $C$ . If  $\text{dom}_{\kappa} F(\kappa) = \text{cod}_{\kappa} G(\kappa)$ , then  $F \cdot G$  is an injections family into  $b$  on  $I$ .
- (74) Let  $F$  be a function from  $I$  into the morphisms of  $C$ . Then  $F$  is a projections family from  $c$  onto  $I$  if and only if  $F^{\text{op}}$  is an injections family into  $c^{\text{op}}$  on  $I$ .
- (75) Let  $F$  be a function from  $I$  into the morphisms of  $C^{\text{op}}$  and  $c$  be an object of  $C^{\text{op}}$ . Then  $F$  is an injections family into  $c$  on  $I$  if and only if  ${}^{\text{op}}F$  is a projections family from  ${}^{\text{op}}c$  onto  $I$ .
- (76) For every injections family  $F$  into  $\text{dom } f$  on  $I$  holds  $F^{\text{op}} \cdot f^{\text{op}} = (f \cdot F)^{\text{op}}$ .

Let us consider  $C, c, I$  and let  $F$  be a function from  $I$  into the morphisms of  $C$ . We say that  $c$  is a coproduct w.r.t.  $F$  if and only if the conditions (Def. 18) are satisfied.

(Def. 18)(i)  $F$  is an injections family into  $c$  on  $I$ , and

- (ii) for every  $d$  and for every injections family  $F'$  into  $d$  on  $I$  such that  $\text{dom}_{\kappa} F(\kappa) = \text{dom}_{\kappa} F'(\kappa)$  there exists  $h$  such that  $h \in \text{hom}(c, d)$  and for every  $k$  such that  $k \in \text{hom}(c, d)$  holds for every  $x$  such that  $x \in I$  holds  $k \cdot F_x = F'_x \cdot h$  iff  $h = k$ .

Next we state several propositions:

- (77) Let  $F$  be a function from  $I$  into the morphisms of  $C$ . Then  $c$  is a product w.r.t.  $F$  if and only if  $c^{\text{op}}$  is a coproduct w.r.t.  $F^{\text{op}}$ .
- (78) Let  $F$  be an injections family into  $c$  on  $I$  and  $F'$  be an injections family into  $d$  on  $I$ . Suppose  $c$  is a coproduct w.r.t.  $F$  and  $d$  is a coproduct w.r.t.  $F'$  and  $\text{dom}_{\kappa} F(\kappa) = \text{dom}_{\kappa} F'(\kappa)$ . Then  $c$  and  $d$  are isomorphic.
- (79) Let  $F$  be an injections family into  $c$  on  $I$ . Suppose  $c$  is a coproduct w.r.t.  $F$  and for all  $x_1, x_2$  such that  $x_1 \in I$  and  $x_2 \in I$  holds  $\text{hom}(\text{dom}(F_{x_1}), \text{dom}(F_{x_2})) \neq \emptyset$ . Let given  $x$ . If  $x \in I$ , then  $F_x$  is coretraction.
- (80) Let  $F$  be an injections family into  $a$  on  $I$ . Suppose  $a$  is a coproduct w.r.t.  $F$  and  $\text{dom } f = a$  and  $\text{cod } f = b$  and  $f$  is invertible. Then  $b$  is a coproduct w.r.t.  $f \cdot F$ .
- (81) For every injections family  $F$  into  $a$  on  $\emptyset$  holds  $a$  is a coproduct w.r.t.  $F$  iff  $a$  is initial.
- (82)  $a$  is a coproduct w.r.t.  $\{y\} \mapsto \text{id}_a$ .
- (83) Let  $F$  be an injections family into  $a$  on  $I$ . Suppose  $a$  is a coproduct w.r.t.  $F$  and for every  $x$  such that  $x \in I$  holds  $\text{dom}(F_x)$  is initial. Then  $a$  is initial.

<sup>4</sup> The proposition (71) has been removed.

Let us consider  $C$ ,  $c$ ,  $i_1$ ,  $i_2$ . We say that  $c$  is a coproduct w.r.t.  $i_1$  and  $i_2$  if and only if the conditions (Def. 19) are satisfied.

- (Def. 19)(i)  $\text{cod } i_1 = c$ ,  
 (ii)  $\text{cod } i_2 = c$ , and  
 (iii) for all  $d, f, g$  such that  $f \in \text{hom}(\text{dom } i_1, d)$  and  $g \in \text{hom}(\text{dom } i_2, d)$  there exists  $h$  such that  $h \in \text{hom}(c, d)$  and for every  $k$  such that  $k \in \text{hom}(c, d)$  holds  $k \cdot i_1 = f$  and  $k \cdot i_2 = g$  iff  $h = k$ .

We now state several propositions:

- (84)  $c$  is a product w.r.t.  $p_1$  and  $p_2$  iff  $c^{\text{op}}$  is a coproduct w.r.t.  $p_1^{\text{op}}$  and  $p_2^{\text{op}}$ .  
 (85) If  $x_1 \neq x_2$ , then  $c$  is a coproduct w.r.t.  $i_1$  and  $i_2$  iff  $c$  is a coproduct w.r.t.  $[x_1 \mapsto i_1, x_2 \mapsto i_2]$ .  
 (86) Suppose  $c$  is a coproduct w.r.t.  $i_1$  and  $i_2$  and  $d$  is a coproduct w.r.t.  $j_1$  and  $j_2$  and  $\text{dom } i_1 = \text{dom } j_1$  and  $\text{dom } i_2 = \text{dom } j_2$ . Then  $c$  and  $d$  are isomorphic.  
 (87) Suppose  $\text{hom}(a, c) \neq \emptyset$  and  $\text{hom}(b, c) \neq \emptyset$ . Let  $i_1$  be a morphism from  $a$  to  $c$  and  $i_2$  be a morphism from  $b$  to  $c$ . Then  $c$  is a coproduct w.r.t.  $i_1$  and  $i_2$  if and only if for every  $d$  such that  $\text{hom}(a, d) \neq \emptyset$  and  $\text{hom}(b, d) \neq \emptyset$  holds  $\text{hom}(c, d) \neq \emptyset$  and for every morphism  $f$  from  $a$  to  $d$  and for every morphism  $g$  from  $b$  to  $d$  there exists a morphism  $h$  from  $c$  to  $d$  such that for every morphism  $k$  from  $c$  to  $d$  holds  $k \cdot i_1 = f$  and  $k \cdot i_2 = g$  iff  $h = k$ .  
 (88) If  $c$  is a coproduct w.r.t.  $i_1$  and  $i_2$  and  $\text{hom}(\text{dom } i_1, \text{dom } i_2) \neq \emptyset$  and  $\text{hom}(\text{dom } i_2, \text{dom } i_1) \neq \emptyset$ , then  $i_1$  is coretraction and  $i_2$  is coretraction.  
 (89) If  $c$  is a coproduct w.r.t.  $i_1$  and  $i_2$  and  $h \in \text{hom}(c, c)$  and  $h \cdot i_1 = i_1$  and  $h \cdot i_2 = i_2$ , then  $h = \text{id}_c$ .  
 (90) Suppose  $c$  is a coproduct w.r.t.  $i_1$  and  $i_2$  and  $\text{dom } f = c$  and  $\text{cod } f = d$  and  $f$  is invertible. Then  $d$  is a coproduct w.r.t.  $f \cdot i_1$  and  $f \cdot i_2$ .  
 (91) If  $c$  is a coproduct w.r.t.  $i_1$  and  $i_2$  and  $\text{dom } i_2$  is initial, then  $\text{dom } i_1$  and  $c$  are isomorphic.  
 (92) If  $c$  is a coproduct w.r.t.  $i_1$  and  $i_2$  and  $\text{dom } i_1$  is initial, then  $\text{dom } i_2$  and  $c$  are isomorphic.

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