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On Powers of Cardinals

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Summary. In the first section the results of [18, axiom (30)][1], i.e. the correspondence between natural and ordinal (cardinal) numbers are shown. The next section is concerned with the concepts of infinity and cofinality (see [8]), and introduces alephs as infinite cardinal numbers. The arithmetics of alephs, i.e. some facts about addition and multiplication, is present in the third section. The concepts of regular and irregular alephs are introduced in the fourth section, and the fact that $\aleph_0$ and every non-limit cardinal number are regular is proved there. Finally, for every alephs α and β

$$\alpha^\beta = \begin{cases} 2^\beta, & \text{if } \alpha \leq \beta, \\ \sum_{\gamma<\alpha} \gamma^\beta, & \text{if } \beta < \mathrm{cf}\alpha \text{ and } \alpha \text{ is limit cardinal}, \\ \left(\sum_{\gamma<\alpha} \gamma^\beta\right)^{\mathrm{cf}\alpha}, & \text{if } \mathrm{cf}\alpha \leq \beta \leq \alpha. \end{cases}$$

Some proofs are based on [16].

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WWW: http://mizar.org/JFM/Vol4/card_5.html

The articles [19], [14], [20], [2], [21], [12], [11], [15], [3], [13], [5], [6], [4], [1], [7], [17], [10], [9], and [8] provide the notation and terminology for this paper.

1. Results of [18, axiom (30)]

For simplicity, we adopt the following convention: n is a natural number, A, B are ordinal numbers, X is a set, and x, y are sets.

One can prove the following propositions:

(1) $1 = \{0\}$ and $2 = \{0,1\}$.

(8)[2] $\mathrm{Seg}\, n = (n+1) \setminus \{0\}$.

2. Infinity, alephs and cofinality

We adopt the following convention: f is a function, K, M, N are cardinal numbers, and p_1, p_2 are sequences of ordinal numbers.

Next we state several propositions:

(9) $\overline{\overline{X}}^+ = X^+$.

[1]Axiom (30) — $n = \{k \in \mathbb{N} : k < n\}$ for every natural number n.
[2] The propositions (2)–(7) have been removed.

(10) $y \in \bigcup f$ iff there exists x such that $x \in \mathrm{dom}\, f$ and $y \in f(x)$.

(11) $\aleph_A$ is infinite.

(12) If M is infinite, then there exists A such that $M = \aleph_A$.

(13) There exists n such that $M = \overline{\overline{n}}$ or there exists A such that $M = \aleph_A$.

Let us consider p_1. Observe that $\bigcup p_1$ is ordinal.
We now state a number of propositions:

(14) Suppose $X \subseteq A$. Then there exists p_1 such that $p_1 =$ the canonical isomorphism between $\subseteq_{\overline{\subseteq_X}}$ and $\subseteq_X$ and p_1 is increasing and $\mathrm{dom}\, p_1 = \overline{\subseteq_X}$ and $\mathrm{rng}\, p_1 = X$.

(15) If $X \subseteq A$, then $\sup X$ is cofinal with $\overline{\subseteq_X}$.

(16) If $X \subseteq A$, then $\overline{\overline{X}} = \overline{\overline{\overline{\subseteq_X}}}$.

(17) There exists B such that $B \subseteq \overline{\overline{A}}$ and A is cofinal with B.

(18) There exists M such that $M \leq \overline{\overline{A}}$ and A is cofinal with M and for every B such that A is cofinal with B holds $M \subseteq B$.

(19) If $\mathrm{rng}\, p_1 = \mathrm{rng}\, p_2$ and p_1 is increasing and p_2 is increasing, then $p_1 = p_2$.

(20) If p_1 is increasing, then p_1 is one-to-one.

(21) $(p_1 \frown p_2) \restriction \mathrm{dom}\, p_1 = p_1$.

(22) If $X \neq \emptyset$, then $\overline{\overline{\{Y; Y \text{ ranges over elements of } 2^X \colon \overline{\overline{Y}} < M\}}} \leq M \cdot \overline{\overline{X}}^M$.

(23) $M < 2^M$.

Let us observe that there exists a set which is infinite and there exists a cardinal number which is infinite.

One can verify that every set which is infinite is also non empty.

An aleph is an infinite cardinal number. Let us consider M. The functor $\mathrm{cf}\, M$ yielding a cardinal number is defined by:

(Def. 2)[3] M is cofinal with $\mathrm{cf}\, M$ and for every N such that M is cofinal with N holds $\mathrm{cf}\, M \leq N$.

Let us consider N. The functor $(\alpha \mapsto \alpha^N)_{\alpha \in M}$ yielding a function yielding cardinal numbers is defined by the conditions (Def. 3).

(Def. 3)(i) For every x holds $x \in \mathrm{dom}((\alpha \mapsto \alpha^N)_{\alpha \in M})$ iff $x \in M$ and x is a cardinal number, and

(ii) for every K such that $K \in M$ holds $(\alpha \mapsto \alpha^N)_{\alpha \in M}(K) = K^N$.

Let us consider A. One can check that $\aleph_A$ is infinite.

3. Arithmetics of alephs

In the sequel a, b are alephs.

The following propositions are true:

(24) There exists A such that $a = \aleph_A$.

(25) $a \neq 0$ and $a \neq 1$ and $a \neq 2$ and $a \neq \overline{\overline{n}}$ and $\overline{\overline{n}} < a$ and $\aleph_0 \leq a$.

(26) If $a \leq M$ or $a < M$, then M is an aleph.

[3] The definition (Def. 1) has been removed.

(27) If $a \leq M$ or $a < M$, then $a + M = M$ and $M + a = M$ and $a \cdot M = M$ and $M \cdot a = M$.

(28) $a + a = a$ and $a \cdot a = a$.

(31)[4] $M \leq M^a$.

(32) $\bigcup a = a$.

Let us consider a, M. Observe that $a + M$ is infinite.
Let us consider M, a. One can check that $M + a$ is infinite.
Let us consider a, b. Observe that $a \cdot b$ is infinite and a^b is infinite.

4. Regular alephs

Let I_1 be an aleph. We say that I_1 is regular if and only if:

(Def. 4) $\operatorname{cf} I_1 = I_1$.

We introduce I_1 is irregular as an antonym of I_1 is regular.
Let us consider a. Observe that a^+ is infinite and every element of a is ordinal.
The following propositions are true:

(34)[5] $\operatorname{cf}(\aleph_0) = \aleph_0$.

(35) $\operatorname{cf}(a^+) = a^+$.

(36) $\aleph_0 \leq \operatorname{cf} a$.

(37) $\operatorname{cf} 0 = 0$ and $\operatorname{cf} \overline{\overline{n+1}} = 1$.

(38) If $X \subseteq M$ and $\overline{\overline{X}} < \operatorname{cf} M$, then $\sup X \in M$ and $\bigcup X \in M$.

(39) If $\operatorname{dom} p_1 = M$ and $\operatorname{rng} p_1 \subseteq N$ and $M < \operatorname{cf} N$, then $\sup p_1 \in N$ and $\bigcup p_1 \in N$.

Let us consider a. Observe that $\operatorname{cf} a$ is infinite.
The following three propositions are true:

(40) If $\operatorname{cf} a < a$, then a is a limit cardinal number.

(41) Suppose $\operatorname{cf} a < a$. Then there exists a sequence x_1 of ordinal numbers such that $\operatorname{dom} x_1 = \operatorname{cf} a$ and $\operatorname{rng} x_1 \subseteq a$ and x_1 is increasing and $a = \sup x_1$ and x_1 is a function yielding cardinal numbers and $0 \notin \operatorname{rng} x_1$.

(42) $\aleph_0$ is regular and a^+ is regular.

5. Infinite powers

In the sequel a, b are alephs.
One can prove the following propositions:

(43) If $a \leq b$, then $a^b = 2^b$.

(44) $(a^+)^b = a^b \cdot a^+$.

(45) $\sum((\alpha \mapsto \alpha^b)_{\alpha \in a}) \leq a^b$.

(46) If a is a limit cardinal number and $b < \operatorname{cf} a$, then $a^b = \sum((\alpha \mapsto \alpha^b)_{\alpha \in a})$.

(47) If $\operatorname{cf} a \leq b$ and $b < a$, then $a^b = (\sum((\alpha \mapsto \alpha^b)_{\alpha \in a}))^{\operatorname{cf} a}$.

[4] The propositions (29) and (30) have been removed.
[5] The proposition (33) has been removed.

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