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Countable Sets and Hessenberg's Theorem

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Summary. The concept of countable sets is introduced and there are shown some facts which deal with finite and countable sets. Besides, the article includes theorems and lemmas on the sum and product of infinite cardinals. The most important of them is Hessenberg's theorem which says that for every infinite cardinal **m** the product $\mathbf{m} \cdot \mathbf{m}$ is equal to **m**.

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The articles [15], [10], [18], [17], [2], [19], [8], [7], [12], [3], [5], [4], [16], [1], [6], [11], [9], [13], and [14] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: X, Y, x denote sets, D denotes a non empty set, m, n, n_1, n_2, n_3, m_2, m_1 denote natural numbers, A, B denote ordinal numbers, L, K, M, N denote cardinal numbers, and f denotes a function.

We now state a number of propositions:

(1) X is finite iff $\overline{\overline{X}}$ is finite.

(2) X is finite iff $\overline{\overline{X}} < \aleph_0$.

(3) If X is finite, then $\overline{\overline{X}} \in \aleph_0$ and $\overline{\overline{X}} \in \omega$.

(4) X is finite iff there exists n such that $\overline{\overline{X}} = \overline{\overline{n}}$.

(5) $\mathrm{succ} A \setminus \{A\} = A$.

(6) If $A \approx n$, then $A = n$.

(7) A is finite iff $A \in \omega$.

(8) A is not finite iff $\omega \subseteq A$.

(9) M is finite iff $M \in \aleph_0$.

(11)[1] M is not finite iff $\aleph_0 \subseteq M$.

(13)[2] If N is finite and M is not finite, then $N < M$ and $N \leq M$.

(14) X is not finite iff there exists Y such that $Y \subseteq X$ and $\overline{\overline{Y}} = \aleph_0$.

(15) ω is not finite and $\mathbb{N}$ is not finite.

[1] The proposition (10) has been removed.
[2] The proposition (12) has been removed.

(16) $\aleph_0$ is not finite.

(17) $X = \emptyset$ iff $\overline{\overline{X}} = 0$.

(19)[3] $0 \leq M$.

(20) $\overline{\overline{X}} = \overline{\overline{Y}}$ iff $X^+ = Y^+$.

(21) $M = N$ iff $N^+ = M^+$.

(22) $N < M$ iff $N^+ \leq M$.

(23) $N < M^+$ iff $N \leq M$.

(24) $0 < M$ iff $1 \leq M$.

(25) $1 < M$ iff $2 \leq M$.

(26) If M is finite and if $N \leq M$ or $N < M$, then N is finite.

We now state a number of propositions:

(27) A is a limit ordinal number iff for all B, n such that $B \in A$ holds $B + n \in A$.

(28) $A + \operatorname{succ} n = \operatorname{succ} A + n$ and $A + (n+1) = \operatorname{succ} A + n$.

(29) There exists n such that $A \cdot \operatorname{succ} \mathbf{1} = A + n$.

(30) If A is a limit ordinal number, then $A \cdot \operatorname{succ} \mathbf{1} = A$.

(31) If $\omega \subseteq A$, then $\mathbf{1} + A = A$.

(32) If M is infinite, then M is a limit ordinal number.

(33) If M is not finite, then $M + M = M$.

(34) If M is not finite and if $N \leq M$ or $N < M$, then $M + N = M$ and $N + M = M$.

(35) If X is not finite and if $X \approx Y$ or $Y \approx X$, then $X \cup Y \approx X$ and $\overline{\overline{X \cup Y}} = \overline{\overline{X}}$.

(36) If X is not finite and Y is finite, then $X \cup Y \approx X$ and $\overline{\overline{X \cup Y}} = \overline{\overline{X}}$.

(37) If X is not finite and if $\overline{\overline{Y}} < \overline{\overline{X}}$ or $\overline{\overline{Y}} \leq \overline{\overline{X}}$, then $X \cup Y \approx X$ and $\overline{\overline{X \cup Y}} = \overline{\overline{X}}$.

(38) For all finite cardinal numbers M, N holds $M + N$ is finite.

(39) If M is not finite, then $M + N$ is not finite and $N + M$ is not finite.

(40) For all finite cardinal numbers M, N holds $M \cdot N$ is finite.

(41) If $K < L$ and $M < N$ or $K \leq L$ and $M < N$ or $K < L$ and $M \leq N$ or $K \leq L$ and $M \leq N$, then $K + M \leq L + N$ and $M + K \leq L + N$.

(42) If $M < N$ or $M \leq N$, then $K + M \leq K + N$ and $K + M \leq N + K$ and $M + K \leq K + N$ and $M + K \leq N + K$.

Let us consider X. We say that X is countable if and only if:

(Def. 1) $\overline{\overline{X}} \leq \aleph_0$.

The following propositions are true:

(43) If X is finite, then X is countable.

[3] The proposition (18) has been removed.

(44) ω is countable and $\mathbb{N}$ is countable.

(45) X is countable iff there exists f such that $\mathrm{dom}\, f = \mathbb{N}$ and $X \subseteq \mathrm{rng}\, f$.

(46) If $Y \subseteq X$ and X is countable, then Y is countable.

(47) If X is countable and Y is countable, then $X \cup Y$ is countable.

(48) If X is countable, then $X \cap Y$ is countable and $Y \cap X$ is countable.

(49) If X is countable, then $X \setminus Y$ is countable.

(50) If X is countable and Y is countable, then $X \dot{-} Y$ is countable.

In the sequel r is a real number.
One can prove the following proposition

(51) $r \neq 0$ or $n = 0$ iff $r^n \neq 0$.

Let m, n be natural numbers. Then m^n is a natural number.
We now state a number of propositions:

(52) If $2^{n_1} \cdot (2 \cdot m_1 + 1) = 2^{n_2} \cdot (2 \cdot m_2 + 1)$, then $n_1 = n_2$ and $m_1 = m_2$.

(53) $[:\mathbb{N}, \mathbb{N}:] \approx \mathbb{N}$ and $\overline{\overline{\mathbb{N}}} = \overline{\overline{[:\mathbb{N}, \mathbb{N}:]}}$.

(54) $\aleph_0 \cdot \aleph_0 = \aleph_0$.

(55) If X is countable and Y is countable, then $[:X, Y:]$ is countable.

(56) $D^1 \approx D$ and $\overline{\overline{D^1}} = \overline{\overline{D}}$.

(57) $[:D^n, D^m:] \approx D^{n+m}$ and $\overline{\overline{[:D^n, D^m:]}} = \overline{\overline{D^{n+m}}}$.

(58) If D is countable, then D^n is countable.

(59) If $\overline{\overline{\mathrm{dom}\, f}} \leq M$ and for every x such that $x \in \mathrm{dom}\, f$ holds $\overline{\overline{f(x)}} \leq N$, then $\overline{\overline{\bigcup f}} \leq M \cdot N$.

(60) If $\overline{\overline{X}} \leq M$ and for every Y such that $Y \in X$ holds $\overline{\overline{Y}} \leq N$, then $\overline{\overline{\bigcup X}} \leq M \cdot N$.

(61) For every f such that $\mathrm{dom}\, f$ is countable and for every x such that $x \in \mathrm{dom}\, f$ holds $f(x)$ is countable holds $\bigcup f$ is countable.

(62) If X is countable and for every Y such that $Y \in X$ holds Y is countable, then $\bigcup X$ is countable.

(63) For every f such that $\mathrm{dom}\, f$ is finite and for every x such that $x \in \mathrm{dom}\, f$ holds $f(x)$ is finite holds $\bigcup f$ is finite.

(65)[4] If D is countable, then D^* is countable.

(66) $\aleph_0 \leq \overline{\overline{D^*}}$.

In this article we present several logical schemes. The scheme *FraenCoun1* deals with a unary functor $\mathcal{F}$ yielding a set and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(n) : \mathcal{P}[n]\}$ is countable

for all values of the parameters.

The scheme *FraenCoun2* deals with a binary functor $\mathcal{F}$ yielding a set and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(n_1, n_2) : \mathcal{P}[n_1, n_2]\}$ is countable

for all values of the parameters.

[4] The proposition (64) has been removed.

The scheme *FraenCoun3* deals with a ternary functor $\mathcal{F}$ yielding a set and a ternary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(n_1, n_2, n_3) : \mathcal{P}[n_1, n_2, n_3]\}$ is countable

for all values of the parameters.

The following propositions are true:

(67) $\aleph_0 \cdot \overline{\overline{n}} \leq \aleph_0$ and $\overline{\overline{n}} \cdot \aleph_0 \leq \aleph_0$.

(68) If $K < L$ and $M < N$ or $K \leq L$ and $M < N$ or $K < L$ and $M \leq N$ or $K \leq L$ and $M \leq N$, then $K \cdot M \leq L \cdot N$ and $M \cdot K \leq L \cdot N$.

(69) If $M < N$ or $M \leq N$, then $K \cdot M \leq K \cdot N$ and $K \cdot M \leq N \cdot K$ and $M \cdot K \leq K \cdot N$ and $M \cdot K \leq N \cdot K$.

(70) If $K < L$ and $M < N$ or $K \leq L$ and $M < N$ or $K < L$ and $M \leq N$ or $K \leq L$ and $M \leq N$, then $K = 0$ or $K^M \leq L^N$.

(71) If $M < N$ or $M \leq N$, then $K = 0$ or $K^M \leq K^N$ and $M^K \leq N^K$.

(72) $M \leq M + N$ and $N \leq M + N$.

(73) If $N \neq 0$, then $M \leq M \cdot N$ and $M \leq N \cdot M$.

(74) If $K < L$ and $M < N$, then $K + M < L + N$ and $M + K < L + N$.

(75) If $K + M < K + N$, then $M < N$.

(76) If $\overline{\overline{X}} + \overline{\overline{Y}} = \overline{\overline{X}}$ and $\overline{\overline{Y}} < \overline{\overline{X}}$, then $\overline{\overline{X \setminus Y}} = \overline{\overline{X}}$.

(77) If M is not finite, then $M \cdot M = M$.

(78) If M is not finite and if $0 < N$ and if $N \leq M$ or $N < M$, then $M \cdot N = M$ and $N \cdot M = M$.

(79) If M is not finite and if $N \leq M$ or $N < M$, then $M \cdot N \leq M$ and $N \cdot M \leq M$.

(80) If X is not finite, then $[:X, X:] \approx X$ and $\overline{\overline{[:X, X:]}} = \overline{\overline{X}}$.

(81) If X is not finite and Y is finite and $Y \neq \emptyset$, then $[:X, Y:] \approx X$ and $\overline{\overline{[:X, Y:]}} = \overline{\overline{X}}$.

(82) If $K < L$ and $M < N$, then $K \cdot M < L \cdot N$ and $M \cdot K < L \cdot N$.

(83) If $K \cdot M < K \cdot N$, then $M < N$.

(84) If X is not finite, then $\overline{\overline{X}} = \aleph_0 \cdot \overline{\overline{X}}$.

(85) If $X \neq \emptyset$ and X is finite and Y is not finite, then $\overline{\overline{Y}} \cdot \overline{\overline{X}} = \overline{\overline{Y}}$.

(86) If D is not finite and $n \neq 0$, then $D^n \approx D$ and $\overline{\overline{D^n}} = \overline{\overline{D}}$.

(87) If D is not finite, then $\overline{\overline{D}} = \overline{\overline{D^*}}$.

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