JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

König's Theorem

Grzegorz Bancerek
Warsaw University
Białystok

Summary. In the article the sum and product of any number of cardinals are introduced and their relationships to addition, multiplication and to other concepts are shown. Then the König's theorem is proved. The theorem that the cardinal of union of increasing family of sets of power less than some cardinal **m** is not greater than **m**, is given too.

MML Identifier: CARD_3.

WWW: http://mizar.org/JFM/Vol2/card_3.html

The articles [11], [7], [14], [13], [15], [5], [6], [2], [12], [1], [10], [8], [9], [3], and [4] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: A, B denote ordinal numbers, K, M, N denote cardinal numbers, x, y, z, X, Y, Z, Z_1, Z_2 denote sets, n denotes a natural number, and f, g denote functions.

Let I_1 be a function. We say that I_1 is cardinal yielding if and only if:

(Def. 1) For every x such that $x \in \operatorname{dom} I_1$ holds $I_1(x)$ is a cardinal number.

Let us note that there exists a function which is cardinal yielding.

A function yielding cardinal numbers is a cardinal yielding function.

In the sequel f_1 denotes a function yielding cardinal numbers.

Let us consider f_1, X. Observe that $f_1 \restriction X$ is cardinal yielding.

Let us consider X, K. Note that $X \longmapsto K$ is cardinal yielding.

The following proposition is true

(3)[1] $\emptyset$ is a function yielding cardinal numbers.

The scheme *CF Lambda* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a cardinal number, and states that:

There exists f_1 such that $\operatorname{dom} f_1 = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds $f_1(x) = \mathcal{F}(x)$

for all values of the parameters.

Let us consider f. The functor $\overline{\overline{f}}$ yielding a function yielding cardinal numbers is defined as follows:

(Def. 2) $\operatorname{dom} \overline{\overline{f}} = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $\overline{\overline{f}}(x) = \overline{\overline{f(x)}}$.

The functor disjoint f yielding a function is defined by:

(Def. 3) $\operatorname{dom} \operatorname{disjoint} f = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $(\operatorname{disjoint} f)(x) = [\!: f(x), \{x\} :\!]$.

[1] The propositions (1) and (2) have been removed.

The functor $\prod f$ yielding a set is defined by:

(Def. 5)[2] $x \in \prod f$ iff there exists g such that $x = g$ and $\mathrm{dom}\, g = \mathrm{dom}\, f$ and for every y such that $y \in \mathrm{dom}\, f$ holds $g(y) \in f(y)$.

The following propositions are true:

(8)[3] $\overline{\overline{f_1}} = f_1$.

(9) $\overline{\overline{\emptyset}} = \emptyset$.

(10) $\overline{\overline{X \longmapsto Y}} = X \longmapsto \overline{\overline{Y}}$.

(11) $\mathrm{disjoint}\, \emptyset = \emptyset$.

(12) $\mathrm{disjoint}(\{x\} \longmapsto X) = \{x\} \longmapsto [\!: X, \{x\} :\!]$.

(13) If $x \in \mathrm{dom}\, f$ and $y \in \mathrm{dom}\, f$ and $x \neq y$, then $(\mathrm{disjoint}\, f)(x)$ misses $(\mathrm{disjoint}\, f)(y)$.

(14) $\bigcup \emptyset = \emptyset$.

(15) $\bigcup (X \longmapsto Y) \subseteq Y$.

(16) If $X \neq \emptyset$, then $\bigcup (X \longmapsto Y) = Y$.

(17) $\bigcup (\{x\} \longmapsto Y) = Y$.

(18) $g \in \prod f$ iff $\mathrm{dom}\, g = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $g(x) \in f(x)$.

(19) $\prod \emptyset = \{\emptyset\}$.

(20) $Y^X = \prod (X \longmapsto Y)$.

Let us consider x, X. The functor $\pi_x X$ yielding a set is defined as follows:

(Def. 6) $y \in \pi_x X$ iff there exists f such that $f \in X$ and $y = f(x)$.

Next we state a number of propositions:

(22)[4] If $x \in \mathrm{dom}\, f$ and $\prod f \neq \emptyset$, then $\pi_x \prod f = f(x)$.

(24)[5] $\pi_x \emptyset = \emptyset$.

(25) $\pi_x \{g\} = \{g(x)\}$.

(26) $\pi_x \{f, g\} = \{f(x), g(x)\}$.

(27) $\pi_x (X \cup Y) = \pi_x X \cup \pi_x Y$.

(28) $\pi_x (X \cap Y) \subseteq \pi_x X \cap \pi_x Y$.

(29) $\pi_x X \setminus \pi_x Y \subseteq \pi_x (X \setminus Y)$.

(30) $\pi_x X \dot{-} \pi_x Y \subseteq \pi_x (X \dot{-} Y)$.

(31) $\overline{\overline{\pi_x X}} \leq \overline{\overline{X}}$.

(32) If $x \in \bigcup \mathrm{disjoint}\, f$, then there exist y, z such that $x = \langle y, z \rangle$.

(33) $x \in \bigcup \mathrm{disjoint}\, f$ iff $x_{\mathbf{2}} \in \mathrm{dom}\, f$ and $x_{\mathbf{1}} \in f(x_{\mathbf{2}})$ and $x = \langle x_{\mathbf{1}}, x_{\mathbf{2}} \rangle$.

(34) If $f \leq g$, then $\mathrm{disjoint}\, f \leq \mathrm{disjoint}\, g$.

---

[2] The definition (Def. 4) has been removed.
[3] The propositions (4)–(7) have been removed.
[4] The proposition (21) has been removed.
[5] The proposition (23) has been removed.

(35) If $f \leq g$, then $\bigcup f \subseteq \bigcup g$.

(36) $\bigcup \mathrm{disjoint}(Y \longmapsto X) = [:X, Y:]$.

(37) $\prod f = \emptyset$ iff $\emptyset \in \mathrm{rng}\, f$.

(38) If $\mathrm{dom}\, f = \mathrm{dom}\, g$ and for every x such that $x \in \mathrm{dom}\, f$ holds $f(x) \subseteq g(x)$, then $\prod f \subseteq \prod g$.

In the sequel F, G are functions yielding cardinal numbers.
One can prove the following two propositions:

(39) For every x such that $x \in \mathrm{dom}\, F$ holds $\overline{\overline{F(x)}} = F(x)$.

(40) For every x such that $x \in \mathrm{dom}\, F$ holds $\overline{\overline{(\mathrm{disjoint}\, F)(x)}} = F(x)$.

Let us consider F. The functor $\sum F$ yielding a cardinal number is defined as follows:

(Def. 7) $\sum F = \overline{\overline{\bigcup \mathrm{disjoint}\, F}}$.

The functor $\prod F$ yields a cardinal number and is defined by:

(Def. 8) $\prod F = \overline{\overline{\prod F}}$.

The following propositions are true:

(43)[6] If $\mathrm{dom}\, F = \mathrm{dom}\, G$ and for every x such that $x \in \mathrm{dom}\, F$ holds $F(x) \subseteq G(x)$, then $\sum F \leq \sum G$.

(44) $\emptyset \in \mathrm{rng}\, F$ iff $\prod F = 0$.

(45) If $\mathrm{dom}\, F = \mathrm{dom}\, G$ and for every x such that $x \in \mathrm{dom}\, F$ holds $F(x) \subseteq G(x)$, then $\prod F \leq \prod G$.

(46) If $F \leq G$, then $\sum F \leq \sum G$.

(47) If $F \leq G$ and $0 \notin \mathrm{rng}\, G$, then $\prod F \leq \prod G$.

(48) $\sum(\emptyset \longmapsto K) = 0$.

(49) $\prod(\emptyset \longmapsto K) = 1$.

(50) $\sum(\{x\} \longmapsto K) = K$.

(51) $\prod(\{x\} \longmapsto K) = K$.

(52) $\sum(M \longmapsto N) = M \cdot N$.

(53) $\prod(N \longmapsto M) = M^N$.

(54) $\overline{\overline{\bigcup f}} \leq \sum \overline{\overline{f}}$.

(55) $\overline{\overline{\bigcup F}} \leq \sum F$.

(56) If $\mathrm{dom}\, F = \mathrm{dom}\, G$ and for every x such that $x \in \mathrm{dom}\, F$ holds $F(x) \in G(x)$, then $\sum F < \prod G$.

Now we present three schemes. The scheme *FinRegularity* deals with a finite set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists x such that $x \in \mathcal{A}$ and for every y such that $y \in \mathcal{A}$ and $y \neq x$ holds not $\mathcal{P}[y, x]$

provided the parameters satisfy the following conditions:

- $\mathcal{A} \neq \emptyset$,
- For all x, y such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, x]$ holds $x = y$, and
- For all x, y, z such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

The scheme *MaxFinSetElem* deals with a finite set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

[6] The propositions (41) and (42) have been removed.

There exists x such that $x \in \mathcal{A}$ and for every y such that $y \in \mathcal{A}$ holds $\mathcal{P}[x,y]$

provided the parameters have the following properties:

- $\mathcal{A} \neq \emptyset$,
- For all x, y holds $\mathcal{P}[x,y]$ or $\mathcal{P}[y,x]$, and
- For all x, y, z such that $\mathcal{P}[x,y]$ and $\mathcal{P}[y,z]$ holds $\mathcal{P}[x,z]$.

The scheme *FuncSeparation* deals with a set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a binary predicate $\mathcal{P}$, and states that:

There exists f such that $\mathrm{dom}\, f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ and for every y holds $y \in f(x)$ iff $y \in \mathcal{F}(x)$ and $\mathcal{P}[x,y]$

for all values of the parameters.

We now state several propositions:

(57) $\mathbf{R}_n$ is finite.

(58) If X is finite, then $\overline{\overline{X}} < \overline{\overline{\omega}}$.

(59) If $\overline{\overline{A}} < \overline{\overline{B}}$, then $A \in B$.

(60) If $\overline{\overline{A}} < M$, then $A \in M$.

(61) Suppose X is $\subseteq$-linear. Then there exists Y such that $Y \subseteq X$ and $\bigcup Y = \bigcup X$ and for every Z such that $Z \subseteq Y$ and $Z \neq \emptyset$ there exists Z_1 such that $Z_1 \in Z$ and for every Z_2 such that $Z_2 \in Z$ holds $Z_1 \subseteq Z_2$.

(62) If for every Z such that $Z \in X$ holds $\overline{\overline{Z}} < M$ and X is $\subseteq$-linear, then $\overline{\overline{\bigcup X}} \leq M$.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/card_1.html.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[3] Grzegorz Bancerek. Cardinal arithmetics. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/card_2.html.

[4] Grzegorz Bancerek. Tarski's classes and ranks. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/classes1.html.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[6] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[7] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[8] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[9] Andrzej Nędzusiak. σ-fields and probability. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/prob_1.html.

[10] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funcop_1.html.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[12] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/mcart_1.html.

[13] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[14] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[15] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received April 10, 1990

Published January 2, 2004