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Properties of the Product of Compact Topological Spaces

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The articles [13], [5], [16], [17], [10], [2], [12], [7], [6], [14], [18], [15], [3], [8], [1], [4], [11], and [9] provide the notation and terminology for this paper.

1. PRELIMINARIES

One can prove the following proposition

(1) For all topological spaces S, T holds $\Omega_{[:S,T:]} = [:\Omega_S, \Omega_T:]$.

Let X be a set and let Y be an empty set. Observe that $[:X, Y:]$ is empty.

Let X be an empty set and let Y be a set. One can check that $[:X, Y:]$ is empty.

We now state the proposition

(2) Let X, Y be non empty topological spaces and x be a point of X. Then $Y \longmapsto x$ is a continuous map from Y into $X \restriction \{x\}$.

Let T be a non empty topological structure. One can verify that id_T is homeomorphism.

Let S, T be non empty topological structures. Let us notice that the predicate S and T are homeomorphic is reflexive and symmetric.

One can prove the following proposition

(3) Let S, T, V be non empty topological spaces. Suppose S and T are homeomorphic and T and V are homeomorphic. Then S and V are homeomorphic.

2. ON THE PROJECTIONS AND EMPTY TOPOLOGICAL SPACES

Let T be a topological structure and let P be an empty subset of T. Note that $T \restriction P$ is empty.

Let us note that there exists a topological space which is strict and empty.

We now state two propositions:

(4) For every topological space T_1 and for every empty topological space T_2 holds $[:T_1, T_2:]$ is empty and $[:T_2, T_1:]$ is empty.

(5) Every empty topological space is compact.

Let us note that every topological space which is empty is also compact.

Let T_1 be a topological space and let T_2 be an empty topological space. One can verify that $[: T_1, T_2 :]$ is empty.

The following propositions are true:

(6) Let X, Y be non empty topological spaces, x be a point of X, and f be a map from $[: Y, X \restriction \{x\} :]$ into Y. If $f = \pi_1((\text{the carrier of } Y) \times \{x\})$, then f is one-to-one.

(7) Let X, Y be non empty topological spaces, x be a point of X, and f be a map from $[: X \restriction \{x\}, Y :]$ into Y. If $f = \pi_2(\{x\} \times \text{the carrier of } Y)$, then f is one-to-one.

(8) Let X, Y be non empty topological spaces, x be a point of X, and f be a map from $[: Y, X \restriction \{x\} :]$ into Y. If $f = \pi_1((\text{the carrier of } Y) \times \{x\})$, then $f^{-1} = \langle \mathrm{id}_Y, Y \longmapsto x \rangle$.

(9) Let X, Y be non empty topological spaces, x be a point of X, and f be a map from $[: X \restriction \{x\}, Y :]$ into Y. If $f = \pi_2(\{x\} \times \text{the carrier of } Y)$, then $f^{-1} = \langle Y \longmapsto x, \mathrm{id}_Y \rangle$.

(10) Let X, Y be non empty topological spaces, x be a point of X, and f be a map from $[: Y, X \restriction \{x\} :]$ into Y. If $f = \pi_1((\text{the carrier of } Y) \times \{x\})$, then f is a homeomorphism.

(11) Let X, Y be non empty topological spaces, x be a point of X, and f be a map from $[: X \restriction \{x\}, Y :]$ into Y. If $f = \pi_2(\{x\} \times \text{the carrier of } Y)$, then f is a homeomorphism.

3. On the Product of Compact Spaces

We now state a number of propositions:

(12) Let X be a non empty topological space, Y be a compact non empty topological space, G be an open subset of $[: X, Y :]$, and x be a set. Suppose $x \in \{x'; x' \text{ ranges over points of } X\colon [: \{x'\}, \text{the carrier of } Y :] \subseteq G\}$. Then there exists a many sorted set f indexed by the carrier of Y such that for every set i if $i \in$ the carrier of Y, then there exists a subset G_1 of X and there exists a subset H_1 of Y such that $f(i) = \langle G_1, H_1 \rangle$ and $\langle x, i \rangle \in [: G_1, H_1 :]$ and G_1 is open and H_1 is open and $[: G_1, H_1 :] \subseteq G$.

(13) Let X be a non empty topological space, Y be a compact non empty topological space, G be an open subset of $[: Y, X :]$, and x be a set. Suppose $x \in \{y; y \text{ ranges over points of } X\colon [: \Omega_Y, \{y\} :] \subseteq G\}$. Then there exists an open subset R of X such that $x \in R$ and $R \subseteq \{y; y \text{ ranges over points of } X\colon [: \Omega_Y, \{y\} :] \subseteq G\}$.

(14) Let X be a non empty topological space, Y be a compact non empty topological space, and G be an open subset of $[: Y, X :]$. Then $\{x; x \text{ ranges over points of } X\colon [: \Omega_Y, \{x\} :] \subseteq G\} \in$ the topology of X.

(15) For all non empty topological spaces X, Y and for every point x of X holds $[: X \restriction \{x\}, Y :]$ and Y are homeomorphic.

(16) For all non empty topological spaces S, T such that S and T are homeomorphic and S is compact holds T is compact.

(17) For all topological spaces X, Y and for every subspace X_1 of X holds $[: Y, X_1 :]$ is a subspace of $[: Y, X :]$.

(18) Let X be a non empty topological space, Y be a compact non empty topological space, x be a point of X, and Z be a subset of $[: Y, X :]$. If $Z = [: \Omega_Y, \{x\} :]$, then Z is compact.

(19) Let X be a non empty topological space, Y be a compact non empty topological space, and x be a point of X. Then $[: Y, X \restriction \{x\} :]$ is compact.

(20) Let X, Y be compact non empty topological spaces and R be a family of subsets of X. Suppose $R = \{Q; Q \text{ ranges over open subsets of } X\colon [: \Omega_Y, Q :] \subseteq \bigcup \mathrm{BaseAppr}(\Omega_{[: Y, X :]})\}$. Then R is open and a cover of Ω_X.

(21) Let X, Y be compact non empty topological spaces, R be a family of subsets of X, and F be a family of subsets of $[:Y, X:]$. Suppose that

(i) F is a cover of $[:Y, X:]$ and open, and

(ii) $R = \{Q; Q$ ranges over open subsets of X: $\bigvee_{F_1 : \text{family of subsets of } [:Y,X:]} (F_1 \subseteq F \wedge F_1$ is finite $\wedge [:\Omega_Y, Q:] \subseteq \bigcup F_1)\}$.

Then R is open and a cover of X.

(22) Let X, Y be compact non empty topological spaces, R be a family of subsets of X, and F be a family of subsets of $[:Y, X:]$. Suppose that

(i) F is a cover of $[:Y, X:]$ and open, and

(ii) $R = \{Q; Q$ ranges over open subsets of X: $\bigvee_{F_1 : \text{family of subsets of } [:Y,X:]} (F_1 \subseteq F \wedge F_1$ is finite $\wedge [:\Omega_Y, Q:] \subseteq \bigcup F_1)\}$.

Then there exists a family C of subsets of X such that $C \subseteq R$ and C is finite and a cover of X.

(23) Let X, Y be compact non empty topological spaces and F be a family of subsets of $[:Y, X:]$. Suppose F is a cover of $[:Y, X:]$ and open. Then there exists a family G of subsets of $[:Y, X:]$ such that $G \subseteq F$ and G is a cover of $[:Y, X:]$ and finite.

(24) For all topological spaces T_1, T_2 such that T_1 is compact and T_2 is compact holds $[:T_1, T_2:]$ is compact.

Let T_1, T_2 be compact topological spaces. Note that $[:T_1, T_2:]$ is compact.
We now state two propositions:

(25) Let X, Y be non empty topological spaces, X_1 be a non empty subspace of X, and Y_1 be a non empty subspace of Y. Then $[:X_1, Y_1:]$ is a subspace of $[:X, Y:]$.

(26) Let X, Y be non empty topological spaces, Z be a non empty subset of $[:Y, X:]$, V be a non empty subset of X, and W be a non empty subset of Y. Suppose $Z = [:W, V:]$. Then the topological structure of $[:Y \restriction W, X \restriction V:] =$ the topological structure of $[:Y, X:] \restriction Z$.

Let T be a topological space. Note that there exists a subset of T which is compact.
Let T be a topological space and let P be a compact subset of T. One can verify that $T \restriction P$ is compact.
One can prove the following proposition

(27) Let T_1, T_2 be topological spaces, S_1 be a subset of T_1, and S_2 be a subset of T_2. If S_1 is compact and S_2 is compact, then $[:S_1, S_2:]$ is a compact subset of $[:T_1, T_2:]$.

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