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Boolean Properties of Lattices

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The article [1] provides the notation and terminology for this paper.

1. General lattice

We adopt the following convention: L is a lattice and X, Y, Z, V are elements of L.

Let us consider L, X, Y. The functor $X \setminus Y$ yields an element of L and is defined by:

(Def. 1) $X \setminus Y = X \sqcap Y^{\rm c}$.

Let us consider L, X, Y. The functor $X \dot{-} Y$ yields an element of L and is defined by:

(Def. 2) $X \dot{-} Y = (X \setminus Y) \sqcup (Y \setminus X)$.

Let us consider L, X, Y. Let us observe that $X = Y$ if and only if:

(Def. 3) $X \sqsubseteq Y$ and $Y \sqsubseteq X$.

Let us consider L, X, Y. We say that X meets Y if and only if:

(Def. 4) $X \sqcap Y \neq \bot_L$.

We introduce X misses Y as an antonym of X meets Y.

We now state a number of propositions:

(3)[1] If $X \sqcup Y \sqsubseteq Z$, then $X \sqsubseteq Z$ and $Y \sqsubseteq Z$.

(4) $X \sqcap Y \sqsubseteq X \sqcup Z$.

(6)[2] If $X \sqsubseteq Z$, then $X \setminus Y \sqsubseteq Z$.

(7) If $X \sqsubseteq Y$, then $X \setminus Z \sqsubseteq Y \setminus Z$.

(8) $X \setminus Y \sqsubseteq X$.

(9) $X \setminus Y \sqsubseteq X \dot{-} Y$.

(10) If $X \setminus Y \sqsubseteq Z$ and $Y \setminus X \sqsubseteq Z$, then $X \dot{-} Y \sqsubseteq Z$.

(11) $X = Y \sqcup Z$ iff $Y \sqsubseteq X$ and $Z \sqsubseteq X$ and for every V such that $Y \sqsubseteq V$ and $Z \sqsubseteq V$ holds $X \sqsubseteq V$.

[1] The propositions (1) and (2) have been removed.
[2] The proposition (5) has been removed.

(12) $X = Y \sqcap Z$ iff $X \sqsubseteq Y$ and $X \sqsubseteq Z$ and for every V such that $V \sqsubseteq Y$ and $V \sqsubseteq Z$ holds $V \sqsubseteq X$.

(14)[3] $X \sqcap (Y \setminus Z) = (X \sqcap Y) \setminus Z$.

(15) If X meets Y, then Y meets X.

(16) X meets X iff $X \neq \perp_L$.

(17) $X \dot{-} Y = Y \dot{-} X$.

Let us consider L, X, Y. Let us note that the predicate X meets Y is symmetric. Let us note that the functor $X \dot{-} Y$ is commutative.

2. Modular lattice

In the sequel L denotes a modular lattice and X, Y denote elements of L.

Next we state the proposition

(21)[4] If X misses Y, then Y misses X.

3. Distributive lattice

In the sequel L is a distributive lattice and X, Y, Z are elements of L.

The following two propositions are true:

(22) If $(X \sqcap Y) \sqcup (X \sqcap Z) = X$, then $X \sqsubseteq Y \sqcup Z$.

(24)[5] $(X \sqcup Y) \setminus Z = (X \setminus Z) \sqcup (Y \setminus Z)$.

4. Distributive lower bounded lattice

In the sequel L is a lower bound lattice and X, Y, Z are elements of L.

Next we state a number of propositions:

(25) If $X \sqsubseteq \perp_L$, then $X = \perp_L$.

(26) If $X \sqsubseteq Y$ and $X \sqsubseteq Z$ and $Y \sqcap Z = \perp_L$, then $X = \perp_L$.

(27) $X \sqcup Y = \perp_L$ iff $X = \perp_L$ and $Y = \perp_L$.

(28) If $X \sqsubseteq Y$ and $Y \sqcap Z = \perp_L$, then $X \sqcap Z = \perp_L$.

(29) $\perp_L \setminus X = \perp_L$.

(30) If X meets Y and $Y \sqsubseteq Z$, then X meets Z.

(31) If X meets $Y \sqcap Z$, then X meets Y and X meets Z.

(32) If X meets $Y \setminus Z$, then X meets Y.

(33) X misses $\perp_L$.

(34) If X misses Z and $Y \sqsubseteq Z$, then X misses Y.

(35) If X misses Y or X misses Z, then X misses $Y \sqcap Z$.

(36) If $X \sqsubseteq Y$ and $X \sqsubseteq Z$ and Y misses Z, then $X = \perp_L$.

(37) If X misses Y, then $Z \sqcap X$ misses $Z \sqcap Y$ and $X \sqcap Z$ misses $Y \sqcap Z$.

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[3] The proposition (13) has been removed.
[4] The propositions (18)–(20) have been removed.
[5] The proposition (23) has been removed.

5. Boolean Lattice

We use the following convention: L denotes a Boolean lattice and X, Y, Z, V denote elements of L.

Next we state a number of propositions:

(38) If $X \setminus Y \sqsubseteq Z$, then $X \sqsubseteq Y \sqcup Z$.

(39) If $X \sqsubseteq Y$, then $Z \setminus Y \sqsubseteq Z \setminus X$.

(40) If $X \sqsubseteq Y$ and $Z \sqsubseteq V$, then $X \setminus V \sqsubseteq Y \setminus Z$.

(41) If $X \sqsubseteq Y \sqcup Z$, then $X \setminus Y \sqsubseteq Z$ and $X \setminus Z \sqsubseteq Y$.

(42) $X^{\rm c} \sqsubseteq (X \sqcap Y)^{\rm c}$ and $Y^{\rm c} \sqsubseteq (X \sqcap Y)^{\rm c}$.

(43) $(X \sqcup Y)^{\rm c} \sqsubseteq X^{\rm c}$ and $(X \sqcup Y)^{\rm c} \sqsubseteq Y^{\rm c}$.

(44) If $X \sqsubseteq Y \setminus X$, then $X = \bot_L$.

(45) If $X \sqsubseteq Y$, then $Y = X \sqcup (Y \setminus X)$.

(46) $X \setminus Y = \bot_L$ iff $X \sqsubseteq Y$.

(47) If $X \sqsubseteq Y \sqcup Z$ and $X \sqcap Z = \bot_L$, then $X \sqsubseteq Y$.

(48) $X \sqcup Y = (X \setminus Y) \sqcup Y$.

(49) $X \setminus (X \sqcup Y) = \bot_L$.

(50) $X \setminus (X \sqcap Y) = X \setminus Y$ and $X \setminus (Y \sqcap X) = X \setminus Y$.

(51) $(X \setminus Y) \sqcap Y = \bot_L$.

(52) $X \sqcup (Y \setminus X) = X \sqcup Y$ and $(Y \setminus X) \sqcup X = Y \sqcup X$.

(53) $(X \sqcap Y) \sqcup (X \setminus Y) = X$.

(54) $X \setminus (Y \setminus Z) = (X \setminus Y) \sqcup (X \sqcap Z)$.

(55) $X \setminus (X \setminus Y) = X \sqcap Y$.

(56) $(X \sqcup Y) \setminus Y = X \setminus Y$.

(57) $X \sqcap Y = \bot_L$ iff $X \setminus Y = X$.

(58) $X \setminus (Y \sqcup Z) = (X \setminus Y) \sqcap (X \setminus Z)$.

(59) $X \setminus (Y \sqcap Z) = (X \setminus Y) \sqcup (X \setminus Z)$.

(60) $X \sqcap (Y \setminus Z) = (X \sqcap Y) \setminus (X \sqcap Z)$ and $(Y \setminus Z) \sqcap X = (Y \sqcap X) \setminus (Z \sqcap X)$.

(61) $(X \sqcup Y) \setminus (X \sqcap Y) = (X \setminus Y) \sqcup (Y \setminus X)$.

(62) $X \setminus Y \setminus Z = X \setminus (Y \sqcup Z)$.

(63) If $X \setminus Y = Y \setminus X$, then $X = Y$.

(66)[6] $X \setminus X = \bot_L$.

(67) $X \setminus \bot_L = X$.

(68) $(X \setminus Y)^{\rm c} = X^{\rm c} \sqcup Y$.

(69) X meets $Y \sqcup Z$ iff X meets Y or X meets Z.

[6] The propositions (64) and (65) have been removed.

(70) $X \sqcap Y$ misses $X \setminus Y$.

(71) X misses $Y \sqcup Z$ iff X misses Y and X misses Z.

(72) $X \setminus Y$ misses Y.

(73) If X misses Y, then $(X \sqcup Y) \setminus Y = X$ and $(X \sqcup Y) \setminus X = Y$.

(74) If $X^{\rm c} \sqcup Y^{\rm c} = X \sqcup Y$ and X misses $X^{\rm c}$ and Y misses $Y^{\rm c}$, then $X = Y^{\rm c}$ and $Y = X^{\rm c}$.

(75) If $X^{\rm c} \sqcup Y^{\rm c} = X \sqcup Y$ and Y misses $X^{\rm c}$ and X misses $Y^{\rm c}$, then $X = X^{\rm c}$ and $Y = Y^{\rm c}$.

(76) $X \dot{-} \bot_L = X$.

(77) $X \dot{-} X = \bot_L$.

(78) $X \sqcap Y$ misses $X \dot{-} Y$.

(79) $X \sqcup Y = X \dot{-} (Y \setminus X)$.

(80) $X \dot{-} (X \sqcap Y) = X \setminus Y$.

(81) $X \sqcup Y = (X \dot{-} Y) \sqcup (X \sqcap Y)$.

(82) $X \dot{-} Y \dot{-} (X \sqcap Y) = X \sqcup Y$.

(83) $X \dot{-} Y \dot{-} (X \sqcup Y) = X \sqcap Y$.

(84) $X \dot{-} Y = (X \sqcup Y) \setminus (X \sqcap Y)$.

(85) $(X \dot{-} Y) \setminus Z = (X \setminus (Y \sqcup Z)) \sqcup (Y \setminus (X \sqcup Z))$.

(86) $X \setminus (Y \dot{-} Z) = (X \setminus (Y \sqcup Z)) \sqcup (X \sqcap Y \sqcap Z)$.

(87) $(X \dot{-} Y) \dot{-} Z = X \dot{-} (Y \dot{-} Z)$.

(88) $(X \dot{-} Y)^{\rm c} = (X \sqcap Y) \sqcup (X^{\rm c} \sqcap Y^{\rm c})$.

REFERENCES

[1] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/lattices.html`.

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