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A Representation of Integers by Binary Arithmetics and Addition of Integers

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Summary. In this article, we introduce the new concept of 2's complement representation. Natural numbers that are congruent mod n can be represented by the same n bits binary. Using the concept introduced here, negative numbers that are congruent mod n also can be represented by the same n bit binary. We also show some properties of addition of integers using this concept.

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The articles [16], [21], [2], [5], [12], [11], [10], [9], [17], [13], [15], [6], [7], [1], [14], [18], [3], [20], [8], [4], and [19] provide the notation and terminology for this paper.

1. Preliminaries

We use the following convention: n denotes a non empty natural number, j, k, l, m denote natural numbers, and g, h, i denote integers.

We now state a number of propositions:

(1) If $m > 0$, then $m \cdot 2 \geq m + 1$.

(2) For every natural number m holds $2^m \geq m$.

(3) For every natural number m holds $\langle \underbrace{0, \ldots, 0}_{m} \rangle + \langle \underbrace{0, \ldots, 0}_{m} \rangle = \langle \underbrace{0, \ldots, 0}_{m} \rangle$.

(4) For every natural number k such that $k \leq l$ and $l \leq m$ holds $k = l$ or $k + 1 \leq l$ and $l \leq m$.

(5) For every non empty natural number n and for all n-tuples x, y of *Boolean* such that $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$ and $y = \langle \underbrace{0, \ldots, 0}_{n} \rangle$ holds $\mathrm{carry}(x, y) = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(6) For every non empty natural number n and for all n-tuples x, y of *Boolean* such that $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$ and $y = \langle \underbrace{0, \ldots, 0}_{n} \rangle$ holds $x + y = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(7) For every non empty natural number n and for every n-tuple F of *Boolean* such that $F = \langle \underbrace{0, \ldots, 0}_{n} \rangle$ holds $\mathrm{Intval}(F) = 0$.

(8) If $l + m \leq k - 1$, then $l < k$ and $m < k$.

(9) If $g \leq h+i$ and $h < 0$ and $i < 0$, then $g < h$ and $g < i$.

(10) If $l+m \leq 2^n - 1$, then $\text{add\_ovfl}(n\text{-BinarySequence}(l), n\text{-BinarySequence}(m)) = \mathit{false}$.

(11) For every non empty natural number n and for all natural numbers l, m such that $l+m \leq 2^n - 1$ holds $\text{Absval}((n\text{-BinarySequence}(l)) + (n\text{-BinarySequence}(m))) = l+m$.

(12) For every non empty natural number n and for every n-tuple z of *Boolean* such that $z_n = \mathit{true}$ holds $\text{Absval}(z) \geq 2^{n-'1}$.

(13) If $l+m \leq 2^{n-'1} - 1$, then $(\text{carry}(n\text{-BinarySequence}(l), n\text{-BinarySequence}(m)))_n = \mathit{false}$.

(14) For every non empty natural number n such that $l+m \leq 2^{n-'1} - 1$ holds $\text{Intval}((n\text{-BinarySequence}(l)) + (n\text{-BinarySequence}(m))) = l+m$.

(15) For every 1-tuple z of *Boolean* such that $z = \langle \mathit{true} \rangle$ holds $\text{Intval}(z) = -1$.

(16) For every 1-tuple z of *Boolean* such that $z = \langle \mathit{false} \rangle$ holds $\text{Intval}(z) = 0$.

(17) For every boolean set x holds $\mathit{true} \vee x = \mathit{true}$.

(18) For every non empty natural number n holds $0 \leq 2^{n-'1} - 1$ and $-2^{n-'1} \leq 0$.

(19) For all n-tuples x, y of *Boolean* such that $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$ and $y = \langle \underbrace{0, \ldots, 0}_{n} \rangle$ holds x and y are summable.

(20) $i \cdot n \bmod n = 0$.

2. MAJORANT POWER

Let m, j be natural numbers. The functor $\text{MajP}(m, j)$ yields a natural number and is defined by:

(Def. 1) $2^{\text{MajP}(m,j)} \geq j$ and $\text{MajP}(m, j) \geq m$ and for every natural number k such that $2^k \geq j$ and $k \geq m$ holds $k \geq \text{MajP}(m, j)$.

We now state several propositions:

(21) If $j \geq k$, then $\text{MajP}(m, j) \geq \text{MajP}(m, k)$.

(22) If $l \geq m$, then $\text{MajP}(l, j) \geq \text{MajP}(m, j)$.

(23) If $m \geq 1$, then $\text{MajP}(m, 1) = m$.

(24) If $j \leq 2^m$, then $\text{MajP}(m, j) = m$.

(25) If $j > 2^m$, then $\text{MajP}(m, j) > m$.

3. 2'S COMPLEMENT

Let m be a natural number and let i be an integer. The functor $\text{2sComplement}(m, i)$ yields a m-tuple of *Boolean* and is defined as follows:

(Def. 2) $\text{2sComplement}(m, i) = \begin{cases} m\text{-BinarySequence}(|2^{\text{MajP}(m,|i|)} + i|), & \text{if } i < 0, \\ m\text{-BinarySequence}(|i|), & \text{otherwise.} \end{cases}$

The following propositions are true:

(26) For every natural number m holds $\text{2sComplement}(m, 0) = \langle \underbrace{0, \ldots, 0}_{m} \rangle$.

(27) For every integer i such that $i \leq 2^{n-'1} - 1$ and $-2^{n-'1} \leq i$ holds $\text{Intval}(\text{2sComplement}(n, i)) = i$.

(28) For all integers h, i such that $h \geq 0$ and $i \geq 0$ or $h < 0$ and $i < 0$ but $h \bmod 2^n = i \bmod 2^n$ holds $2\mathrm{sComplement}(n,h) = 2\mathrm{sComplement}(n,i)$.

(29) For all integers h, i such that $h \geq 0$ and $i \geq 0$ or $h < 0$ and $i < 0$ but $h \equiv i (\bmod 2^n)$ holds $2\mathrm{sComplement}(n,h) = 2\mathrm{sComplement}(n,i)$.

(30) For all natural numbers l, m such that $l \bmod 2^n = m \bmod 2^n$ holds n-BinarySequence$(l) =$ n-BinarySequence(m).

(31) For all natural numbers l, m such that $l \equiv m (\bmod 2^n)$ holds n-BinarySequence$(l) =$ n-BinarySequence(m).

(32) For every natural number j such that $1 \leq j$ and $j \leq n$ holds $(2\mathrm{sComplement}(n+1,i))_j = (2\mathrm{sComplement}(n,i))_j$.

(33) There exists an element x of *Boolean* such that $2\mathrm{sComplement}(m+1,i) = (2\mathrm{sComplement}(m,i)) ^\frown \langle x \rangle$.

(34) There exists an element x of *Boolean* such that $(m+1)$-BinarySequence$(l) =$ $(m$-BinarySequence$(l)) ^\frown \langle x \rangle$.

(35) Let n be a non empty natural number. Suppose $-2^n \leq h+i$ and $h < 0$ and $i < 0$ and $-2^{n-'1} \leq h$ and $-2^{n-'1} \leq i$. Then $(\mathrm{carry}(2\mathrm{sComplement}(n+1,h), 2\mathrm{sComplement}(n+1,i)))_{n+1} = \mathit{true}$.

(36) For every non empty natural number n such that $-2^{n-'1} \leq h+i$ and $h+i \leq 2^{n-'1}-1$ and $h \geq 0$ and $i \geq 0$ holds $\mathrm{Intval}(2\mathrm{sComplement}(n,h) + 2\mathrm{sComplement}(n,i)) = h+i$.

(37) Let n be a non empty natural number. Suppose $-2^{(n+1)-'1} \leq h+i$ and $h+i \leq 2^{(n+1)-'1}-1$ and $h < 0$ and $i < 0$ and $-2^{n-'1} \leq h$ and $-2^{n-'1} \leq i$. Then $\mathrm{Intval}(2\mathrm{sComplement}(n+1,h) + 2\mathrm{sComplement}(n+1,i)) = h+i$.

(38) Let n be a non empty natural number. Suppose that $-2^{n-'1} \leq h$ and $h \leq 2^{n-'1}-1$ and $-2^{n-'1} \leq i$ and $i \leq 2^{n-'1}-1$ and $-2^{n-'1} \leq h+i$ and $h+i \leq 2^{n-'1}-1$ and $h \geq 0$ and $i < 0$ or $h < 0$ and $i \geq 0$. Then $\mathrm{Intval}(2\mathrm{sComplement}(n,h) + 2\mathrm{sComplement}(n,i)) = h+i$.

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