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Series in Banach and Hilbert Spaces

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Summary. In [14] the series of real numbers were investigated. The introduction to Banach and Hilbert spaces ([10], [11],[12]), enables us to arrive at the concept of series in Hilbert space. We start with the notions: partial sums of series, sum and *n*-th sum of series, convergent series (summable series), absolutely convergent series. We prove some basic theorems: the necessary condition for a series to converge, Weierstrass' test, d'Alembert's test, Cauchy's test.

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The articles [17], [2], [15], [4], [1], [3], [7], [5], [6], [14], [8], [16], [9], [10], [11], [12], and [13] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: X is a real unitary space, a, b, r are real numbers, s_1, s_2, s_3 are sequences of X, R_1, R_2, R_3 are sequences of real numbers, and k, n, m are natural numbers.

The scheme *Rec Func Ex RUS* deals with a real unitary space $\mathcal{A}$, a point $\mathcal{B}$ of $\mathcal{A}$, and a binary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

> There exists a function f from $\mathbb{N}$ into the carrier of $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and for every element n of $\mathbb{N}$ and for every point x of $\mathcal{A}$ such that $x = f(n)$ holds $f(n+1) = \mathcal{F}(n,x)$

for all values of the parameters.

Let us consider X and let us consider s_1. The functor $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ yields a sequence of X and is defined by:

(Def. 1) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(0) = s_1(0)$ and for every n holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) + s_1(n+1)$.

The following propositions are true:

(1) $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_2+s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

(2) $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_2-s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

(3) $(\sum_{\alpha=0}^{\kappa}(a\cdot s_1)(\alpha))_{\kappa\in\mathbb{N}} = a\cdot(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$.

(4) $(\sum_{\alpha=0}^{\kappa}(-s_1)(\alpha))_{\kappa\in\mathbb{N}} = -(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$.

(5) $a\cdot(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} + b\cdot(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(a\cdot s_2+b\cdot s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

Let us consider X and let us consider s_1. We say that s_1 is summable if and only if:

(Def. 2) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent.

The functor $\sum s_1$ yielding a point of X is defined as follows:

(Def. 3) $\sum s_1 = \lim((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})$.

Next we state several propositions:

(6) If s_2 is summable and s_3 is summable, then s_2+s_3 is summable and $\sum(s_2+s_3) = \sum s_2 + \sum s_3$.

(7) If s_2 is summable and s_3 is summable, then s_2-s_3 is summable and $\sum(s_2-s_3) = \sum s_2 - \sum s_3$.

(8) If s_1 is summable, then $a\cdot s_1$ is summable and $\sum(a\cdot s_1) = a\cdot\sum s_1$.

(9) If s_1 is summable, then s_1 is convergent and $\lim s_1 = 0_X$.

(10) Suppose X is a Hilbert space. Then s_1 is summable if and only if for every r such that $r>0$ there exists k such that for all n, m such that $n\geq k$ and $m\geq k$ holds $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m)\| < r$.

(11) If s_1 is summable, then $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ is bounded.

(12) For all s_1, s_2 such that for every n holds $s_2(n) = s_1(0)$ holds $(\sum_{\alpha=0}^{\kappa}(s_1\uparrow 1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}\uparrow 1 - s_2$.

(13) If s_1 is summable, then for every k holds $s_1\uparrow k$ is summable.

(14) If there exists k such that $s_1\uparrow k$ is summable, then s_1 is summable.

Let us consider X, s_1, n. The functor $\sum_{\kappa=0}^{n} s_1(\kappa)$ yielding a point of X is defined by:

(Def. 4) $\sum_{\kappa=0}^{n} s_1(\kappa) = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)$.

We now state several propositions:

(16)[1] $\sum_{\kappa=0}^{0} s_1(\kappa) = s_1(0)$.

(17) $\sum_{\kappa=0}^{1} s_1(\kappa) = \sum_{\kappa=0}^{0} s_1(\kappa) + s_1(1)$.

(18) $\sum_{\kappa=0}^{1} s_1(\kappa) = s_1(0) + s_1(1)$.

(19) $\sum_{\kappa=0}^{n+1} s_1(\kappa) = \sum_{\kappa=0}^{n} s_1(\kappa) + s_1(n+1)$.

(20) $s_1(n+1) = \sum_{\kappa=0}^{n+1} s_1(\kappa) - \sum_{\kappa=0}^{n} s_1(\kappa)$.

(21) $s_1(1) = \sum_{\kappa=0}^{1} s_1(\kappa) - \sum_{\kappa=0}^{0} s_1(\kappa)$.

Let us consider X, s_1, n, m. The functor $\sum_{\kappa=n+1}^{m} s_1(\kappa)$ yields a point of X and is defined as follows:

(Def. 5) $\sum_{\kappa=n+1}^{m} s_1(\kappa) = \sum_{\kappa=0}^{n} s_1(\kappa) - \sum_{\kappa=0}^{m} s_1(\kappa)$.

One can prove the following propositions:

(23)[2] $\sum_{\kappa=1+1}^{0} s_1(\kappa) = s_1(1)$.

(24) $\sum_{\kappa=n+1+1}^{n} s_1(\kappa) = s_1(n+1)$.

(25) Suppose X is a Hilbert space. Then s_1 is summable if and only if for every r such that $r>0$ there exists k such that for all n, m such that $n\geq k$ and $m\geq k$ holds $\|\sum_{\kappa=0}^{n} s_1(\kappa) - \sum_{\kappa=0}^{m} s_1(\kappa)\| < r$.

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[1] The proposition (15) has been removed.
[2] The proposition (22) has been removed.

(26) Suppose X is a Hilbert space. Then s_1 is summable if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geq k$ and $m \geq k$ holds $\|\sum_{\kappa=n+1}^{m} s_1(\kappa)\| < r$.

Let us consider R_1, n. The functor $\sum_{\kappa=0}^{n} R_1(\kappa)$ yielding a real number is defined by:

(Def. 6) $\sum_{\kappa=0}^{n} R_1(\kappa) = (\sum_{\alpha=0}^{\kappa}(R_1)(\alpha))_{\kappa\in\mathbb{N}}(n)$.

Let us consider R_1, n, m. The functor $\sum_{\kappa=n+1}^{m} R_1(\kappa)$ yielding a real number is defined as follows:

(Def. 7) $\sum_{\kappa=n+1}^{m} R_1(\kappa) = \sum_{\kappa=0}^{n} R_1(\kappa) - \sum_{\kappa=0}^{m} R_1(\kappa)$.

Let us consider X, s_1. We say that s_1 is absolutely summable if and only if:

(Def. 8) $\|s_1\|$ is summable.

Next we state a number of propositions:

(27) If s_2 is absolutely summable and s_3 is absolutely summable, then $s_2 + s_3$ is absolutely summable.

(28) If s_1 is absolutely summable, then $a \cdot s_1$ is absolutely summable.

(29) If for every n holds $\|s_1\|(n) \leq R_1(n)$ and R_1 is summable, then s_1 is absolutely summable.

(30) If for every n holds $s_1(n) \neq 0_X$ and $R_1(n) = \frac{\|s_1(n+1)\|}{\|s_1(n)\|}$ and R_1 is convergent and $\lim R_1 < 1$, then s_1 is absolutely summable.

(31) If $r > 0$ and there exists m such that for every n such that $n \geq m$ holds $\|s_1(n)\| \geq r$, then s_1 is not convergent or $\lim s_1 \neq 0_X$.

(32) If for every n holds $s_1(n) \neq 0_X$ and there exists m such that for every n such that $n \geq m$ holds $\frac{\|s_1(n+1)\|}{\|s_1(n)\|} \geq 1$, then s_1 is not summable.

(33) If for every n holds $s_1(n) \neq 0_X$ and for every n holds $R_1(n) = \frac{\|s_1(n+1)\|}{\|s_1(n)\|}$ and R_1 is convergent and $\lim R_1 > 1$, then s_1 is not summable.

(34) If for every n holds $R_1(n) = \sqrt[n]{\|s_1(n)\|}$ and R_1 is convergent and $\lim R_1 < 1$, then s_1 is absolutely summable.

(35) If for every n holds $R_1(n) = \sqrt[n]{\|s_1\|(n)}$ and there exists m such that for every n such that $n \geq m$ holds $R_1(n) \geq 1$, then s_1 is not summable.

(36) If for every n holds $R_1(n) = \sqrt[n]{\|s_1\|(n)}$ and R_1 is convergent and $\lim R_1 > 1$, then s_1 is not summable.

(37) $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}$ is non-decreasing.

(38) For every n holds $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(n) \geq 0$.

(39) For every n holds $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)\| \leq (\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(40) For every n holds $\|\sum_{\kappa=0}^{n} s_1(\kappa)\| \leq \sum_{\kappa=0}^{n}\|s_1\|(\kappa)$.

(41) For all n, m holds $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)\| \leq |(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(n)|$.

(42) For all n, m holds $\|\sum_{\kappa=0}^{m} s_1(\kappa) - \sum_{\kappa=0}^{n} s_1(\kappa)\| \leq |\sum_{\kappa=0}^{m}\|s_1\|(\kappa) - \sum_{\kappa=0}^{n}\|s_1\|(\kappa)|$.

(43) For all n, m holds $\|\sum_{\kappa=m+1}^{n} s_1(\kappa)\| \leq |\sum_{\kappa=m+1}^{n}\|s_1\|(\kappa)|$.

(44) If X is a Hilbert space, then if s_1 is absolutely summable, then s_1 is summable.

Let us consider X, s_1, R_1. The functor $R_1 \cdot s_1$ yielding a sequence of X is defined by:

(Def. 9) For every n holds $(R_1 \cdot s_1)(n) = R_1(n) \cdot s_1(n)$.

The following propositions are true:

(45) $R_1 \cdot (s_2 + s_3) = R_1 \cdot s_2 + R_1 \cdot s_3$.

(46) $(R_2 + R_3) \cdot s_1 = R_2 \cdot s_1 + R_3 \cdot s_1$.

(47) $(R_2\, R_3) \cdot s_1 = R_2 \cdot (R_3 \cdot s_1)$.

(48) $(a\, R_1) \cdot s_1 = a \cdot (R_1 \cdot s_1)$.

(49) $R_1 \cdot -s_1 = (-R_1) \cdot s_1$.

(50) If R_1 is convergent and s_1 is convergent, then $R_1 \cdot s_1$ is convergent.

(51) If R_1 is bounded and s_1 is bounded, then $R_1 \cdot s_1$ is bounded.

(52) If R_1 is convergent and s_1 is convergent, then $R_1 \cdot s_1$ is convergent and $\lim(R_1 \cdot s_1) = \lim R_1 \cdot \lim s_1$.

Let us consider R_1. We say that R_1 is Cauchy if and only if:

(Def. 10) For every r such that $r > 0$ there exists k such that for all n, m such that $n \geq k$ and $m \geq k$ holds $|R_1(n) - R_1(m)| < r$.

We introduce R_1 is a Cauchy sequence as a synonym of R_1 is Cauchy.

Next we state four propositions:

(53) If X is a Hilbert space, then if s_1 is a Cauchy sequence and R_1 is a Cauchy sequence, then $R_1 \cdot s_1$ is a Cauchy sequence.

(54) For every n holds $(\sum_{\alpha=0}^{\kappa}((R_1 - R_1 \uparrow 1) \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n) = (\sum_{\alpha=0}^{\kappa}(R_1 \cdot s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) - (R_1 \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(n+1)$.

(55) For every n holds $(\sum_{\alpha=0}^{\kappa}(R_1 \cdot s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (R_1 \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(n+1) - (\sum_{\alpha=0}^{\kappa}((R_1 \uparrow 1 - R_1) \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(56) For every n holds $\sum_{\kappa=0}^{n+1}(R_1 \cdot s_1)(\kappa) = (R_1 \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(n+1) - \sum_{\kappa=0}^{n}((R_1 \uparrow 1 - R_1) \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(\kappa)$.

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