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Introduction to Banach and Hilbert Spaces — Part II

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Summary. A continuation of [6]. It contains the definitions of the convergent sequence and limit of the sequence. The convergence with respect to the norm and the distance is also introduced. Last part of this article is devoted to the following concepts: ball, closed ball and sphere.

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The articles [7], [1], [8], [2], [4], [3], [10], [9], [6], and [5] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: X is a real unitary space, x, y, z, g, g_1, g_2 are points of X, a, q, r are real numbers, s_1, s_2, s_3, s'_1 are sequences of X, and k, n, m are natural numbers.

Let us consider X, s_1. We say that s_1 is convergent if and only if:

(Def. 1) There exists g such that for every r such that $r > 0$ there exists m such that for every n such that $n \geq m$ holds $\rho(s_1(n), g) < r$.

One can prove the following propositions:

(1) If s_1 is constant, then s_1 is convergent.

(2) If s_1 is convergent and there exists k such that for every n such that $k \leq n$ holds $s'_1(n) = s_1(n)$, then s'_1 is convergent.

(3) If s_2 is convergent and s_3 is convergent, then $s_2 + s_3$ is convergent.

(4) If s_2 is convergent and s_3 is convergent, then $s_2 - s_3$ is convergent.

(5) If s_1 is convergent, then $a \cdot s_1$ is convergent.

(6) If s_1 is convergent, then $-s_1$ is convergent.

(7) If s_1 is convergent, then $s_1 + x$ is convergent.

(8) If s_1 is convergent, then $s_1 - x$ is convergent.

(9) s_1 is convergent if and only if there exists g such that for every r such that $r > 0$ there exists m such that for every n such that $n \geq m$ holds $\|s_1(n) - g\| < r$.

Let us consider X, s_1. Let us assume that s_1 is convergent. The functor $\lim s_1$ yields a point of X and is defined by:

(Def. 2) For every r such that $r > 0$ there exists m such that for every n such that $n \geq m$ holds $\rho(s_1(n), \lim s_1) < r$.

One can prove the following propositions:

(10) If s_1 is constant and $x \in \operatorname{rng} s_1$, then $\lim s_1 = x$.

(11) If s_1 is constant and there exists n such that $s_1(n) = x$, then $\lim s_1 = x$.

(12) If s_1 is convergent and there exists k such that for every n such that $n \geq k$ holds $s'_1(n) = s_1(n)$, then $\lim s_1 = \lim s'_1$.

(13) If s_2 is convergent and s_3 is convergent, then $\lim(s_2 + s_3) = \lim s_2 + \lim s_3$.

(14) If s_2 is convergent and s_3 is convergent, then $\lim(s_2 - s_3) = \lim s_2 - \lim s_3$.

(15) If s_1 is convergent, then $\lim(a \cdot s_1) = a \cdot \lim s_1$.

(16) If s_1 is convergent, then $\lim(-s_1) = -\lim s_1$.

(17) If s_1 is convergent, then $\lim(s_1 + x) = \lim s_1 + x$.

(18) If s_1 is convergent, then $\lim(s_1 - x) = \lim s_1 - x$.

(19) Suppose s_1 is convergent. Then $\lim s_1 = g$ if and only if for every r such that $r > 0$ there exists m such that for every n such that $n \geq m$ holds $\|s_1(n) - g\| < r$.

Let us consider X, s_1. The functor $\|s_1\|$ yields a sequence of real numbers and is defined as follows:

(Def. 3) For every n holds $\|s_1\|(n) = \|s_1(n)\|$.

We now state three propositions:

(20) If s_1 is convergent, then $\|s_1\|$ is convergent.

(21) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1\|$ is convergent and $\lim\|s_1\| = \|g\|$.

(22) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - g\|$ is convergent and $\lim\|s_1 - g\| = 0$.

Let us consider X, let us consider s_1, and let us consider x. The functor $\rho(s_1, x)$ yielding a sequence of real numbers is defined as follows:

(Def. 4) For every n holds $(\rho(s_1, x))(n) = \rho(s_1(n), x)$.

We now state a number of propositions:

(23) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1, g)$ is convergent.

(24) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1, g)$ is convergent and $\lim \rho(s_1, g) = 0$.

(25) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 + s_3\|$ is convergent and $\lim\|s_2 + s_3\| = \|g_1 + g_2\|$.

(26) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|(s_2 + s_3) - (g_1 + g_2)\|$ is convergent and $\lim\|(s_2 + s_3) - (g_1 + g_2)\| = 0$.

(27) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 - s_3\|$ is convergent and $\lim\|s_2 - s_3\| = \|g_1 - g_2\|$.

(28) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 - s_3 - (g_1 - g_2)\|$ is convergent and $\lim\|s_2 - s_3 - (g_1 - g_2)\| = 0$.

(29) If s_1 is convergent and $\lim s_1 = g$, then $\|a \cdot s_1\|$ is convergent and $\lim\|a \cdot s_1\| = \|a \cdot g\|$.

(30) If s_1 is convergent and $\lim s_1 = g$, then $\|a \cdot s_1 - a \cdot g\|$ is convergent and $\lim\|a \cdot s_1 - a \cdot g\| = 0$.

(31) If s_1 is convergent and $\lim s_1 = g$, then $\|-s_1\|$ is convergent and $\lim\|-s_1\| = \|-g\|$.

(32) If s_1 is convergent and $\lim s_1 = g$, then $\|-s_1 - -g\|$ is convergent and $\lim\|-s_1 - -g\| = 0$.

(33) If s_1 is convergent and $\lim s_1 = g$, then $\|(s_1 + x) - (g + x)\|$ is convergent and $\lim\|(s_1 + x) - (g + x)\| = 0$.

(34) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - x\|$ is convergent and $\lim\|s_1 - x\| = \|g - x\|$.

(35) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - x - (g - x)\|$ is convergent and $\lim\|s_1 - x - (g - x)\| = 0$.

(36) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\rho(s_2 + s_3, g_1 + g_2)$ is convergent and $\lim \rho(s_2 + s_3, g_1 + g_2) = 0$.

(37) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\rho(s_2 - s_3, g_1 - g_2)$ is convergent and $\lim \rho(s_2 - s_3, g_1 - g_2) = 0$.

(38) If s_1 is convergent and $\lim s_1 = g$, then $\rho(a \cdot s_1, a \cdot g)$ is convergent and $\lim \rho(a \cdot s_1, a \cdot g) = 0$.

(39) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1 + x, g + x)$ is convergent and $\lim \rho(s_1 + x, g + x) = 0$.

Let us consider X, x, r. The functor $\mathrm{Ball}(x, r)$ yields a subset of X and is defined by:

(Def. 5) $\mathrm{Ball}(x, r) = \{y; y$ ranges over points of X: $\|x - y\| < r\}$.

The functor $\overline{\mathrm{Ball}}(x, r)$ yields a subset of X and is defined by:

(Def. 6) $\overline{\mathrm{Ball}}(x, r) = \{y; y$ ranges over points of X: $\|x - y\| \leq r\}$.

The functor $\mathrm{Sphere}(x, r)$ yields a subset of X and is defined by:

(Def. 7) $\mathrm{Sphere}(x, r) = \{y; y$ ranges over points of X: $\|x - y\| = r\}$.

The following propositions are true:

(40) $z \in \mathrm{Ball}(x, r)$ iff $\|x - z\| < r$.

(41) $z \in \mathrm{Ball}(x, r)$ iff $\rho(x, z) < r$.

(42) If $r > 0$, then $x \in \mathrm{Ball}(x, r)$.

(43) If $y \in \mathrm{Ball}(x, r)$ and $z \in \mathrm{Ball}(x, r)$, then $\rho(y, z) < 2 \cdot r$.

(44) If $y \in \mathrm{Ball}(x, r)$, then $y - z \in \mathrm{Ball}(x - z, r)$.

(45) If $y \in \mathrm{Ball}(x, r)$, then $y - x \in \mathrm{Ball}(0_X, r)$.

(46) If $y \in \mathrm{Ball}(x, r)$ and $r \leq q$, then $y \in \mathrm{Ball}(x, q)$.

(47) $z \in \overline{\mathrm{Ball}}(x, r)$ iff $\|x - z\| \leq r$.

(48) $z \in \overline{\mathrm{Ball}}(x, r)$ iff $\rho(x, z) \leq r$.

(49) If $r \geq 0$, then $x \in \overline{\mathrm{Ball}}(x, r)$.

(50) If $y \in \mathrm{Ball}(x, r)$, then $y \in \overline{\mathrm{Ball}}(x, r)$.

(51) $z \in \mathrm{Sphere}(x, r)$ iff $\|x - z\| = r$.

(52) $z \in \mathrm{Sphere}(x, r)$ iff $\rho(x, z) = r$.

(53) If $y \in \mathrm{Sphere}(x, r)$, then $y \in \overline{\mathrm{Ball}}(x, r)$.

(54) $\mathrm{Ball}(x, r) \subseteq \overline{\mathrm{Ball}}(x, r)$.

(55) $\mathrm{Sphere}(x, r) \subseteq \overline{\mathrm{Ball}}(x, r)$.

(56) $\mathrm{Ball}(x, r) \cup \mathrm{Sphere}(x, r) = \overline{\mathrm{Ball}}(x, r)$.

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