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Introduction to Banach and Hilbert Spaces — Part I

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Summary. Basing on the notion of real linear space (see [11]) we introduce real unitary space. At first, we define the scalar product of two vectors and examine some of its properties. On the base of this notion we introduce the norm and the distance in real unitary space and study properties of these concepts. Next, proceeding from the definition of the sequence in real unitary space and basic operations on sequences we prove several theorems which will be used in our further considerations.

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The articles [4], [12], [1], [9], [5], [2], [3], [13], [8], [6], [11], [10], and [7] provide the notation and terminology for this paper.

We consider unitary space structures as extensions of RLS structure as systems

⟨ a carrier, a zero, an addition, an external multiplication, a scalar product ⟩,

where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, the external multiplication is a function from $[:\mathbb{R}, \text{the carrier}:]$ into the carrier, and the scalar product is a function from $[:\text{the carrier}, \text{the carrier}:]$ into $\mathbb{R}$.

Let us note that there exists a unitary space structure which is non empty and strict.

Let D be a non empty set, let Z be an element of D, let a be a binary operation on D, let m be a function from $[:\mathbb{R}, D:]$ into D, and let s be a function from $[:D, D:]$ into $\mathbb{R}$. One can check that $\langle D, Z, a, m, s \rangle$ is non empty.

We follow the rules: X is a non empty unitary space structure, a, b are real numbers, and x, y are points of X.

Let us consider X and let us consider x, y. The functor $(x|y)$ yields a real number and is defined by:

(Def. 1) $(x|y) = (\text{the scalar product of } X)(\langle x, y \rangle)$.

Let I_1 be a non empty unitary space structure. We say that I_1 is real unitary space-like if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let x, y, z be points of I_1 and given a. Then $(x|x) = 0$ iff $x = 0_{(I_1)}$ and $0 \leq (x|x)$ and $(x|y) = (y|x)$ and $((x+y)|z) = (x|z) + (y|z)$ and $((a \cdot x)|y) = a \cdot (x|y)$.

Let us note that there exists a non empty unitary space structure which is real unitary space-like, real linear space-like, Abelian, add-associative, right zeroed, right complementable, and strict.

A real unitary space is a real unitary space-like real linear space-like Abelian add-associative right zeroed right complementable non empty unitary space structure.

We use the following convention: X denotes a real unitary space and x, y, z, u, v denote points of X.

Let us consider X and let us consider x, y. Let us note that the functor $(x|y)$ is commutative.

We now state a number of propositions:

(6)[1] $(0_X|0_X) = 0$.

(7) $(x|(y+z)) = (x|y)+(x|z)$.

(8) $(x|(a\cdot y)) = a\cdot(x|y)$.

(9) $((a\cdot x)|y) = (x|(a\cdot y))$.

(10) $((a\cdot x+b\cdot y)|z) = a\cdot(x|z)+b\cdot(y|z)$.

(11) $(x|(a\cdot y+b\cdot z)) = a\cdot(x|y)+b\cdot(x|z)$.

(12) $((-x)|y) = (x|-y)$.

(13) $((-x)|y) = -(x|y)$.

(14) $(x|-y) = -(x|y)$.

(15) $((-x)|-y) = (x|y)$.

(16) $((x-y)|z) = (x|z)-(y|z)$.

(17) $(x|(y-z)) = (x|y)-(x|z)$.

(18) $((x-y)|(u-v)) = ((x|u)-(x|v)-(y|u))+(y|v)$.

(19) $(0_X|x) = 0$.

(20) $(x|0_X) = 0$.

(21) $((x+y)|(x+y)) = (x|x)+2\cdot(x|y)+(y|y)$.

(22) $((x+y)|(x-y)) = (x|x)-(y|y)$.

(23) $((x-y)|(x-y)) = ((x|x)-2\cdot(x|y))+(y|y)$.

(24) $|(x|y)| \leq \sqrt{(x|x)}\cdot\sqrt{(y|y)}$.

Let us consider X and let us consider x, y. We say that x, y are orthogonal if and only if:

(Def. 3) $(x|y) = 0$.

Let us note that the predicate x, y are orthogonal is symmetric.

Next we state several propositions:

(26)[2] If x, y are orthogonal, then x, $-y$ are orthogonal.

(27) If x, y are orthogonal, then $-x$, y are orthogonal.

(28) If x, y are orthogonal, then $-x$, $-y$ are orthogonal.

(29) x, 0_X are orthogonal.

(30) If x, y are orthogonal, then $((x+y)|(x+y)) = (x|x)+(y|y)$.

(31) If x, y are orthogonal, then $((x-y)|(x-y)) = (x|x)+(y|y)$.

Let us consider X, x. The functor $\|x\|$ yielding a real number is defined as follows:

(Def. 4) $\|x\| = \sqrt{(x|x)}$.

Next we state several propositions:

(32) $\|x\| = 0$ iff $x = 0_X$.

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[1] The propositions (1)–(5) have been removed.

[2] The proposition (25) has been removed.

(33) $\|a \cdot x\| = |a| \cdot \|x\|$.

(34) $0 \leq \|x\|$.

(35) $|(x|y)| \leq \|x\| \cdot \|y\|$.

(36) $\|x+y\| \leq \|x\| + \|y\|$.

(37) $\|-x\| = \|x\|$.

(38) $\|x\| - \|y\| \leq \|x-y\|$.

(39) $|\|x\| - \|y\|| \leq \|x-y\|$.

Let us consider X, x, y. The functor $\rho(x,y)$ yields a real number and is defined by:

(Def. 5) $\rho(x,y) = \|x-y\|$.

One can prove the following proposition

(40) $\rho(x,y) = \rho(y,x)$.

Let us consider X, x, y. Let us observe that the functor $\rho(x,y)$ is commutative.
One can prove the following propositions:

(41) $\rho(x,x) = 0$.

(42) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$.

(43) $x \neq y$ iff $\rho(x,y) \neq 0$.

(44) $\rho(x,y) \geq 0$.

(45) $x \neq y$ iff $\rho(x,y) > 0$.

(46) $\rho(x,y) = \sqrt{((x-y)|(x-y))}$.

(47) $\rho(x+y,u+v) \leq \rho(x,u) + \rho(y,v)$.

(48) $\rho(x-y,u-v) \leq \rho(x,u) + \rho(y,v)$.

(49) $\rho(x-z,y-z) = \rho(x,y)$.

(50) $\rho(x-z,y-z) \leq \rho(z,x) + \rho(z,y)$.

We use the following convention: s_1, s_2, s_3, s_4 denote sequences of X and k, n, m denote natural numbers.

The scheme *Ex Seq in RUS* deals with a non empty unitary space structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

There exists a sequence s_1 of $\mathcal{A}$ such that for every n holds $s_1(n) = \mathcal{F}(n)$

for all values of the parameters.

Let us consider X and let us consider s_1. The functor $-s_1$ yields a sequence of X and is defined by:

(Def. 10)[3] For every n holds $(-s_1)(n) = -s_1(n)$.

Let us consider X, let us consider s_1, and let us consider x. The functor $s_1 + x$ yields a sequence of X and is defined as follows:

(Def. 12)[4] For every n holds $(s_1 + x)(n) = s_1(n) + x$.

We now state the proposition

[3] The definitions (Def. 6)–(Def. 9) have been removed.

[4] The definition (Def. 11) has been removed.

(55)[5] $s_2 + s_3 = s_3 + s_2$.

Let us consider X, s_2, s_3. Let us note that the functor $s_2 + s_3$ is commutative.
We now state a number of propositions:

(56) $s_2 + (s_3 + s_4) = (s_2 + s_3) + s_4$.

(57) If s_2 is constant and s_3 is constant and $s_1 = s_2 + s_3$, then s_1 is constant.

(58) If s_2 is constant and s_3 is constant and $s_1 = s_2 - s_3$, then s_1 is constant.

(59) If s_2 is constant and $s_1 = a \cdot s_2$, then s_1 is constant.

(68)[6] s_1 is constant iff for every n holds $s_1(n) = s_1(n+1)$.

(69) s_1 is constant iff for all n, k holds $s_1(n) = s_1(n+k)$.

(70) s_1 is constant iff for all n, m holds $s_1(n) = s_1(m)$.

(71) $s_2 - s_3 = s_2 + -s_3$.

(72) $s_1 = s_1 + 0_X$.

(73) $a \cdot (s_2 + s_3) = a \cdot s_2 + a \cdot s_3$.

(74) $(a+b) \cdot s_1 = a \cdot s_1 + b \cdot s_1$.

(75) $(a \cdot b) \cdot s_1 = a \cdot (b \cdot s_1)$.

(76) $1 \cdot s_1 = s_1$.

(77) $(-1) \cdot s_1 = -s_1$.

(78) $s_1 - x = s_1 + -x$.

(79) $s_2 - s_3 = -(s_3 - s_2)$.

(80) $s_1 = s_1 - 0_X$.

(81) $s_1 = --s_1$.

(82) $s_2 - (s_3 + s_4) = s_2 - s_3 - s_4$.

(83) $(s_2 + s_3) - s_4 = s_2 + (s_3 - s_4)$.

(84) $s_2 - (s_3 - s_4) = (s_2 - s_3) + s_4$.

(85) $a \cdot (s_2 - s_3) = a \cdot s_2 - a \cdot s_3$.

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[6] The propositions (60)–(67) have been removed.

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