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# Arithmetic of Non-Negative Rational Numbers[1]

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The articles [4], [5], [1], [2], and [3] provide the notation and terminology for this paper.

## 1. NATURAL ORDINALS

In this paper $A$ is an ordinal number.

Let $A$ be an ordinal number. One can verify that every element of $A$ is ordinal.

One can verify the following observations:

* every ordinal number which is empty is also natural,
* **1** is natural and non empty, and
* every element of ω is natural.

Let us observe that there exists an ordinal number which is non empty and natural.

Let $a$ be a natural ordinal number. Note that $\mathrm{succ}\, a$ is natural.

The scheme *Omega Ind* concerns a unary predicate $\mathcal{P}$, and states that:

For every natural ordinal number $a$ holds $\mathcal{P}[a]$

provided the following conditions are met:

- $\mathcal{P}[\emptyset]$, and
- For every natural ordinal number $a$ such that $\mathcal{P}[a]$ holds $\mathcal{P}[\mathrm{succ}\, a]$.

Let $a$, $b$ be natural ordinal numbers. One can verify that $a+b$ is natural.

The following proposition is true

(1) For all ordinal numbers $a$, $b$ such that $a+b$ is natural holds $a \in \omega$ and $b \in \omega$.

Let $a$, $b$ be natural ordinal numbers. Note that $a-b$ is natural and $a \cdot b$ is natural.

The following propositions are true:

(2) For all ordinal numbers $a$, $b$ such that $a \cdot b$ is natural and non empty holds $a \in \omega$ and $b \in \omega$.

(3) For all natural ordinal numbers $a$, $b$ holds $a+b=b+a$.

(4) For all natural ordinal numbers $a$, $b$ holds $a \cdot b = b \cdot a$.

Let $a$, $b$ be natural ordinal numbers. Let us note that the functor $a+b$ is commutative. Let us notice that the functor $a \cdot b$ is commutative.

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## 2. Relative Prime Numbers and Divisibility

Let $a$, $b$ be ordinal numbers. We say that $a$ and $b$ are relative prime if and only if:

(Def. 1) For all ordinal numbers $c$, $d_1$, $d_2$ such that $a = c \cdot d_1$ and $b = c \cdot d_2$ holds $c = \mathbf{1}$.

Let us note that the predicate $a$ and $b$ are relative prime is symmetric.

The following propositions are true:

(5) $\emptyset$ and $\emptyset$ are not relative prime.

(6) $\mathbf{1}$ and $A$ are relative prime.

(7) If $\emptyset$ and $A$ are relative prime, then $A = \mathbf{1}$.

In the sequel $a$, $b$, $c$ denote natural ordinal numbers.

We now state the proposition

(8) Suppose $a \neq \emptyset$ or $b \neq \emptyset$. Then there exist natural ordinal numbers $c$, $d_1$, $d_2$ such that $d_1$ and $d_2$ are relative prime and $a = c \cdot d_1$ and $b = c \cdot d_2$.

In the sequel $l$, $m$, $n$ denote natural ordinal numbers.

Let us consider $m$, $n$. One can check that $m \div n$ is natural and $m \bmod n$ is natural.

Let $k$, $n$ be ordinal numbers. The predicate $k \mid n$ is defined as follows:

(Def. 2) There exists an ordinal number $a$ such that $n = k \cdot a$.

Let us note that the predicate $k \mid n$ is reflexive.

We now state several propositions:

(9) $a \mid b$ iff there exists $c$ such that $b = a \cdot c$.

(10) For all $m$, $n$ such that $\emptyset \in m$ holds $n \bmod m \in m$.

(11) For all $n$, $m$ holds $m \mid n$ iff $n = m \cdot (n \div m)$.

(13)[1] For all $n$, $m$ such that $n \mid m$ and $m \mid n$ holds $n = m$.

(14) $n \mid \emptyset$ and $\mathbf{1} \mid n$.

(15) For all $n$, $m$ such that $\emptyset \in m$ and $n \mid m$ holds $n \subseteq m$.

(16) For all $n$, $m$, $l$ such that $n \mid m$ and $n \mid m + l$ holds $n \mid l$.

Let $k$, $n$ be natural ordinal numbers. The functor $\mathrm{lcm}(k, n)$ yields an element of $\omega$ and is defined as follows:

(Def. 3) $k \mid \mathrm{lcm}(k, n)$ and $n \mid \mathrm{lcm}(k, n)$ and for every $m$ such that $k \mid m$ and $n \mid m$ holds $\mathrm{lcm}(k, n) \mid m$.

Let us observe that the functor $\mathrm{lcm}(k, n)$ is commutative.

The following two propositions are true:

(17) $\mathrm{lcm}(m, n) \mid m \cdot n$.

(18) If $n \neq \emptyset$, then $m \cdot n \div \mathrm{lcm}(m, n) \mid m$.

Let $k$, $n$ be natural ordinal numbers. The functor $\gcd(k, n)$ yields an element of $\omega$ and is defined as follows:

(Def. 4) $\gcd(k, n) \mid k$ and $\gcd(k, n) \mid n$ and for every $m$ such that $m \mid k$ and $m \mid n$ holds $m \mid \gcd(k, n)$.

[1] The proposition (12) has been removed.

Let us observe that the functor $\gcd(k,n)$ is commutative.

The following propositions are true:

(19) $\gcd(a,\emptyset) = a$ and $\mathrm{lcm}(a,\emptyset) = \emptyset$.

(20) If $\gcd(a,b) = \emptyset$, then $a = \emptyset$.

(21) $\gcd(a,a) = a$ and $\mathrm{lcm}(a,a) = a$.

(22) $\gcd(a \cdot c, b \cdot c) = \gcd(a,b) \cdot c$.

(23) If $b \neq \emptyset$, then $\gcd(a,b) \neq \emptyset$ and $b \div \gcd(a,b) \neq \emptyset$.

(24) If $a \neq \emptyset$ or $b \neq \emptyset$, then $a \div \gcd(a,b)$ and $b \div \gcd(a,b)$ are relative prime.

(25) $a$ and $b$ are relative prime iff $\gcd(a,b) = \mathbf{1}$.

Let $a$, $b$ be natural ordinal numbers. The functor $\mathrm{RED}(a,b)$ yielding an element of $\omega$ is defined by:

(Def. 5) $\mathrm{RED}(a,b) = a \div \gcd(a,b)$.

Next we state several propositions:

(26) $\mathrm{RED}(a,b) \cdot \gcd(a,b) = a$.

(27) If $a \neq \emptyset$ or $b \neq \emptyset$, then $\mathrm{RED}(a,b)$ and $\mathrm{RED}(b,a)$ are relative prime.

(28) If $a$ and $b$ are relative prime, then $\mathrm{RED}(a,b) = a$.

(29) $\mathrm{RED}(a,\mathbf{1}) = a$ and $\mathrm{RED}(\mathbf{1},a) = \mathbf{1}$.

(30) If $b \neq \emptyset$, then $\mathrm{RED}(b,a) \neq \emptyset$.

(31) $\mathrm{RED}(\emptyset,a) = \emptyset$ and if $a \neq \emptyset$, then $\mathrm{RED}(a,\emptyset) = \mathbf{1}$.

(32) If $a \neq \emptyset$, then $\mathrm{RED}(a,a) = \mathbf{1}$.

(33) If $c \neq \emptyset$, then $\mathrm{RED}(a \cdot c, b \cdot c) = \mathrm{RED}(a,b)$.

## 3. Non-Negative Rationals

In the sequel $i$, $j$, $k$ are elements of $\omega$.

The functor $\mathbb{Q}_+$ is defined by:

(Def. 6) $\mathbb{Q}_+ = (\{\langle i, j\rangle : i \text{ and } j \text{ are relative prime} \wedge j \neq \emptyset\} \setminus \{\langle k, \mathbf{1}\rangle\}) \cup \omega$.

One can prove the following proposition

(34) $\omega \subseteq \mathbb{Q}_+$.

In the sequel $x$, $y$, $z$ are elements of $\mathbb{Q}_+$.

Let us mention that $\mathbb{Q}_+$ is non empty.

One can check that there exists an element of $\mathbb{Q}_+$ which is non empty and ordinal.

The following propositions are true:

(35) $x \in \omega$ or there exist $i$, $j$ such that $x = \langle i, j\rangle$ and $i$ and $j$ are relative prime and $j \neq \emptyset$ and $j \neq \mathbf{1}$.

(36) It is not true that there exist sets $i$, $j$ such that $\langle i, j\rangle$ is an ordinal number.

(37) If $A \in \mathbb{Q}_+$, then $A \in \omega$.

Let us mention that every ordinal element of $\mathbb{Q}_+$ is natural.
One can prove the following two propositions:

(38) It is not true that there exist sets $i$, $j$ such that $\langle i, j\rangle \in \omega$.

(39) $\langle i, j\rangle \in \mathbb{Q}_+$ iff $i$ and $j$ are relative prime and $j \neq \emptyset$ and $j \neq \mathbf{1}$.

Let $x$ be an element of $\mathbb{Q}_+$. The functor $\mathrm{num}\,x$ yielding an element of $\omega$ is defined as follows:

(Def. 7)(i) $\mathrm{num}\,x = x$ if $x \in \omega$,

(ii) there exists $a$ such that $x = \langle \mathrm{num}\,x, a\rangle$, otherwise.

The functor $\mathrm{den}\,x$ yielding an element of $\omega$ is defined as follows:

(Def. 8)(i) $\mathrm{den}\,x = \mathbf{1}$ if $x \in \omega$,

(ii) there exists $a$ such that $x = \langle a, \mathrm{den}\,x\rangle$, otherwise.

One can prove the following propositions:

(40) $\mathrm{num}\,x$ and $\mathrm{den}\,x$ are relative prime.

(41) $\mathrm{den}\,x \neq \emptyset$.

(42) If $x \notin \omega$, then $x = \langle \mathrm{num}\,x, \mathrm{den}\,x\rangle$ and $\mathrm{den}\,x \neq \mathbf{1}$.

(43) $x \neq \emptyset$ iff $\mathrm{num}\,x \neq \emptyset$.

(44) $x \in \omega$ iff $\mathrm{den}\,x = \mathbf{1}$.

Let $i$, $j$ be natural ordinal numbers. The functor $\frac{i}{j}$ yields an element of $\mathbb{Q}_+$ and is defined by:

(Def. 9) $\frac{i}{j} = \begin{cases} \text{(i)} \quad \emptyset, \text{ if } j = \emptyset, \\ \text{(ii)} \quad \mathrm{RED}(i, j), \text{ if } \mathrm{RED}(j, i) = \mathbf{1}, \\ \langle \mathrm{RED}(i, j), \mathrm{RED}(j, i)\rangle, \text{ otherwise.} \end{cases}$

We introduce $\mathrm{quotient}(i, j)$ as a synonym of $\frac{i}{j}$.
We now state several propositions:

(45) $\frac{\mathrm{num}\,x}{\mathrm{den}\,x} = x$.

(46) $\frac{\emptyset}{b} = \emptyset$ and $\frac{a}{\mathbf{1}} = a$.

(47) If $a \neq \emptyset$, then $\frac{a}{a} = \mathbf{1}$.

(48) If $b \neq \emptyset$, then $\mathrm{num}(\frac{a}{b}) = \mathrm{RED}(a, b)$ and $\mathrm{den}(\frac{a}{b}) = \mathrm{RED}(b, a)$.

(49) If $i$ and $j$ are relative prime and $j \neq \emptyset$, then $\mathrm{num}(\frac{i}{j}) = i$ and $\mathrm{den}(\frac{i}{j}) = j$.

(50) If $c \neq \emptyset$, then $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.

In the sequel $i$, $j$, $k$ are natural ordinal numbers.
We now state the proposition

(51) If $j \neq \emptyset$ and $l \neq \emptyset$, then $\frac{i}{j} = \frac{k}{l}$ iff $i \cdot l = j \cdot k$.

Let $x$, $y$ be elements of $\mathbb{Q}_+$. The functor $x + y$ yields an element of $\mathbb{Q}_+$ and is defined by:

(Def. 10) $x + y = \frac{\mathrm{num}\,x \cdot \mathrm{den}\,y + \mathrm{num}\,y \cdot \mathrm{den}\,x}{\mathrm{den}\,x \cdot \mathrm{den}\,y}$.

Let us observe that the functor $x + y$ is commutative. The functor $x * y$ yielding an element of $\mathbb{Q}_+$ is defined as follows:

(Def. 11) $x * y = \frac{\mathrm{num}\,x \cdot \mathrm{num}\,y}{\mathrm{den}\,x \cdot \mathrm{den}\,y}$.

Let us observe that the functor $x * y$ is commutative.

The following propositions are true:

(52) If $j \neq \emptyset$ and $l \neq \emptyset$, then $\frac{i}{j} + \frac{k}{l} = \frac{i \cdot l + j \cdot k}{j \cdot l}$.

(53) If $k \neq \emptyset$, then $\frac{i}{k} + \frac{j}{k} = \frac{i+j}{k}$.

Let us observe that there exists an element of $\mathbb{Q}_+$ which is empty.

$\emptyset$ is an empty element of $\mathbb{Q}_+$. Then $\mathbf{1}$ is a non empty ordinal element of $\mathbb{Q}_+$.

Next we state a number of propositions:

(54) $x * \emptyset = \emptyset$.

(55) $\frac{i}{j} * \frac{k}{l} = \frac{i \cdot k}{j \cdot l}$.

(56) $x + \emptyset = x$.

(57) $(x+y)+z = x+(y+z)$.

(58) $(x*y)*z = x*(y*z)$.

(59) $x * \mathbf{1} = x$.

(60) If $x \neq \emptyset$, then there exists $y$ such that $x * y = \mathbf{1}$.

(61) If $x \neq \emptyset$, then there exists $z$ such that $y = x * z$.

(62) If $x \neq \emptyset$ and $x * y = x * z$, then $y = z$.

(63) $x * (y+z) = x*y + x*z$.

(64) For all ordinal elements $i$, $j$ of $\mathbb{Q}_+$ holds $i + j = i + j$.

(65) For all ordinal elements $i$, $j$ of $\mathbb{Q}_+$ holds $i * j = i \cdot j$.

(66) There exists $y$ such that $x = y + y$.

Let $x$, $y$ be elements of $\mathbb{Q}_+$. The predicate $x \leq y$ is defined as follows:

(Def. 12) There exists an element $z$ of $\mathbb{Q}_+$ such that $y = x + z$.

Let us note that the predicate $x \leq y$ is connected. We introduce $y < x$ as an antonym of $x \leq y$.

In the sequel $r$, $s$, $t$ denote elements of $\mathbb{Q}_+$.

The following propositions are true:

(68)[2] It is not true that there exists a set $y$ such that $\langle \emptyset, y \rangle \in \mathbb{Q}_+$.

(69) If $s + t = r + t$, then $s = r$.

(70) If $r + s = \emptyset$, then $r = \emptyset$.

(71) $\emptyset \leq s$.

(72) If $s \leq \emptyset$, then $s = \emptyset$.

(73) If $r \leq s$ and $s \leq r$, then $r = s$.

(74) If $r \leq s$ and $s \leq t$, then $r \leq t$.

(75) $r < s$ iff $r \leq s$ and $r \neq s$.

(76) If $r < s$ and $s \leq t$ or $r \leq s$ and $s < t$, then $r < t$.

(77) If $r < s$ and $s < t$, then $r < t$.

[2] The proposition (67) has been removed.

(78) If $x \in \omega$ and $x+y \in \omega$, then $y \in \omega$.

(79) For every ordinal element $i$ of $\mathbb{Q}_+$ such that $i < x$ and $x < i + \mathbf{1}$ holds $x \notin \omega$.

(80) If $t \neq \emptyset$, then there exists $r$ such that $r < t$ and $r \notin \omega$.

(81) $\{s : s < t\} \in \mathbb{Q}_+$ iff $t = \emptyset$.

(82) Let $A$ be a subset of $\mathbb{Q}_+$. Suppose there exists $t$ such that $t \in A$ and $t \neq \emptyset$ and for all $r$, $s$ such that $r \in A$ and $s \leq r$ holds $s \in A$. Then there exist elements $r_1$, $r_2$, $r_3$ of $\mathbb{Q}_+$ such that $r_1 \in A$ and $r_2 \in A$ and $r_3 \in A$ and $r_1 \neq r_2$ and $r_2 \neq r_3$ and $r_3 \neq r_1$.

(83) $s+t \leq r+t$ iff $s \leq r$.

(85)[3] $s \leq s+t$.

(86) If $r * s = \emptyset$, then $r = \emptyset$ or $s = \emptyset$.

(87) If $r \leq s * t$, then there exists an element $t_0$ of $\mathbb{Q}_+$ such that $r = s * t_0$ and $t_0 \leq t$.

(88) If $t \neq \emptyset$ and $s * t \leq r * t$, then $s \leq r$.

(89) For all elements $r_1$, $r_2$, $s_1$, $s_2$ of $\mathbb{Q}_+$ such that $r_1 + r_2 = s_1 + s_2$ holds $r_1 \leq s_1$ or $r_2 \leq s_2$.

(90) If $s \leq r$, then $s * t \leq r * t$.

(91) For all elements $r_1$, $r_2$, $s_1$, $s_2$ of $\mathbb{Q}_+$ such that $r_1 * r_2 = s_1 * s_2$ holds $r_1 \leq s_1$ or $r_2 \leq s_2$.

(92) $r = \emptyset$ iff $r + s = s$.

(93) For all elements $s_1$, $t_1$, $s_2$, $t_2$ of $\mathbb{Q}_+$ such that $s_1 + t_1 = s_2 + t_2$ and $s_1 \leq s_2$ holds $t_2 \leq t_1$.

(94) If $r \leq s$ and $s \leq r+t$, then there exists an element $t_0$ of $\mathbb{Q}_+$ such that $s = r + t_0$ and $t_0 \leq t$.

(95) If $r \leq s+t$, then there exist elements $s_0$, $t_0$ of $\mathbb{Q}_+$ such that $r = s_0 + t_0$ and $s_0 \leq s$ and $t_0 \leq t$.

(96) If $r < s$ and $r < t$, then there exists an element $t_0$ of $\mathbb{Q}_+$ such that $t_0 \leq s$ and $t_0 \leq t$ and $r < t_0$.

(97) If $r \leq s$ and $s \leq t$ and $s \neq t$, then $r \neq t$.

(98) If $s < r+t$ and $t \neq \emptyset$, then there exist elements $r_0$, $t_0$ of $\mathbb{Q}_+$ such that $s = r_0 + t_0$ and $r_0 \leq r$ and $t_0 \leq t$ and $t_0 \neq t$.

(99) For every non empty subset $A$ of $\mathbb{Q}_+$ such that $A \in \mathbb{Q}_+$ there exists $s$ such that $s \in A$ and for every $r$ such that $r \in A$ holds $r \leq s$.

(100) There exists $t$ such that $r + t = s$ or $s + t = r$.

(101) If $r < s$, then there exists $t$ such that $r < t$ and $t < s$.

(102) There exists $s$ such that $r < s$.

(103) If $t \neq \emptyset$, then there exists $s$ such that $s \in \omega$ and $r \leq s * t$.

The scheme *DisNat* deals with elements $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ of $\mathbb{Q}_+$ and a unary predicate $\mathcal{P}$, and states that:

There exists $s$ such that $s \in \omega$ and $\mathcal{P}[s]$ and not $\mathcal{P}[s+\mathcal{B}]$

provided the parameters satisfy the following conditions:

- $\mathcal{B} = \mathbf{1}$,
- $\mathcal{A} = \emptyset$,
- $\mathcal{C} \in \omega$,
- $\mathcal{P}[\mathcal{A}]$, and
- Not $\mathcal{P}[\mathcal{C}]$.

[3] The proposition (84) has been removed.

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