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Non-Negative Real Numbers. Part I[1]

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The articles [5], [4], [6], [1], [2], and [3] provide the notation and terminology for this paper.

In this paper r, s, t, x', y' denote elements of $\mathbb{Q}_+$.

The subset DedekindCuts of $2^{\mathbb{Q}_+}$ is defined as follows:

(Def. 1) $\mathrm{DedekindCuts} = \{A; A$ ranges over subsets of $\mathbb{Q}_+$: $r \in A \Rightarrow \bigwedge_s (s \leq r \Rightarrow s \in A) \wedge \bigvee_s (s \in A \wedge r < s)\} \setminus \{\mathbb{Q}_+\}$.

Let us mention that DedekindCuts is non empty.

The functor $\mathbb{R}_+$ is defined by:

(Def. 2) $\mathbb{R}_+ = (\mathbb{Q}_+ \cup \mathrm{DedekindCuts}) \setminus \{\{s : s < t\} : t \neq \emptyset\}$.

In the sequel x, y, z denote elements of $\mathbb{R}_+$.

The following propositions are true:

(1) $\mathbb{Q}_+ \subseteq \mathbb{R}_+$.

(2) $\omega \subseteq \mathbb{R}_+$.

One can verify that $\mathbb{R}_+$ is non empty.

Let us consider x. The functor DedekindCut x yielding an element of DedekindCuts is defined by:

(Def. 3)(i) There exists r such that $x = r$ and $\mathrm{DedekindCut}\, x = \{s : s < r\}$ if $x \in \mathbb{Q}_+$,

(ii) $\mathrm{DedekindCut}\, x = x$, otherwise.

One can prove the following proposition

(3) It is not true that there exists a set y such that $\langle \emptyset, y \rangle \in \mathbb{R}_+$.

Let x be an element of DedekindCuts. The functor Glued x yields an element of $\mathbb{R}_+$ and is defined as follows:

(Def. 4)(i) There exists r such that $\mathrm{Glued}\, x = r$ and for every s holds $s \in x$ iff $s < r$ if there exists r such that for every s holds $s \in x$ iff $s < r$,

(ii) $\mathrm{Glued}\, x = x$, otherwise.

Let x, y be elements of $\mathbb{R}_+$. The predicate $x \leq y$ is defined by:

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(Def. 5)(i) There exist x', y' such that $x = x'$ and $y = y'$ and $x' \leq y'$ if $x \in \mathbb{Q}_+$ and $y \in \mathbb{Q}_+$,

(ii) $x \in y$ if $x \in \mathbb{Q}_+$ and $y \notin \mathbb{Q}_+$,

(iii) $y \notin x$ if $x \notin \mathbb{Q}_+$ and $y \in \mathbb{Q}_+$,

(iv) $x \subseteq y$, otherwise.

Let us note that the predicate $x \leq y$ is connected. We introduce $y < x$ as an antonym of $x \leq y$.

Let A, B be elements of DedekindCuts. The functor $A + B$ yielding an element of DedekindCuts is defined by:

(Def. 6) $A + B = \{r + s : r \in A \wedge s \in B\}$.

Let us note that the functor $A + B$ is commutative.

Let A, B be elements of DedekindCuts. The functor $A * B$ yields an element of DedekindCuts and is defined by:

(Def. 7) $A * B = \{r * s : r \in A \wedge s \in B\}$.

Let us notice that the functor $A * B$ is commutative.

Let x, y be elements of $\mathbb{R}_+$. The functor $x + y$ yielding an element of $\mathbb{R}_+$ is defined by:

(Def. 8) $x + y = \begin{cases} \text{(i)} \quad x, \text{ if } y = \emptyset, \\ \text{(ii)} \quad y, \text{ if } x = \emptyset, \\ \text{Glued}\,(\text{DedekindCut}\, x + \text{DedekindCut}\, y), \text{ otherwise.} \end{cases}$

Let us notice that the functor $x + y$ is commutative. The functor $x * y$ yielding an element of $\mathbb{R}_+$ is defined by:

(Def. 9) $x * y = \text{Glued}\,(\text{DedekindCut}\, x * \text{DedekindCut}\, y)$.

Let us notice that the functor $x * y$ is commutative.

One can prove the following propositions:

(4) If $x = \emptyset$, then $x * y = \emptyset$.

(6)[1] If $x + y = \emptyset$, then $x = \emptyset$.

(7) $x + (y + z) = (x + y) + z$.

(8) I_1 is $\subseteq$-linear, where $I_1 = \{A; A$ ranges over subsets of $\mathbb{Q}_+$: $r \in A \Rightarrow \bigwedge_s (s \leq r \Rightarrow s \in A) \wedge \bigvee_s (s \in A \wedge r < s)\}$.

(9) Let X, Y be subsets of $\mathbb{R}_+$. Suppose there exists x such that $x \in X$ and there exists x such that $x \in Y$ and for all x, y such that $x \in X$ and $y \in Y$ holds $x \leq y$. Then there exists z such that for all x, y such that $x \in X$ and $y \in Y$ holds $x \leq z$ and $z \leq y$.

(10) If $x \leq y$, then there exists z such that $x + z = y$.

(11) There exists z such that $x + z = y$ or $y + z = x$.

(12) If $x + y = x + z$, then $y = z$.

(13) $x * (y * z) = (x * y) * z$.

(14) $x * (y + z) = x * y + x * z$.

(15) If $x \neq \emptyset$, then there exists y such that $x * y = \mathbf{1}$.

(16) If $x = \mathbf{1}$, then $x * y = y$.

(17) If $x \in \omega$ and $y \in \omega$, then $y + x \in \omega$.

[1] The proposition (5) has been removed.

(18) For every subset A of $\mathbb{R}_+$ such that $\emptyset \in A$ and for all x, y such that $x \in A$ and $y = \mathbf{1}$ holds $x + y \in A$ holds $\omega \subseteq A$.

(19) For every x such that $x \in \omega$ and for every y holds $y \in x$ iff $y \in \omega$ and $y \neq x$ and $y \leq x$.

(20) If $x = y + z$, then $z \leq x$.

(21) $\emptyset \in \mathbb{R}_+$ and $\mathbf{1} \in \mathbb{R}_+$.

(22) If $x \in \mathbb{Q}_+$ and $y \in \mathbb{Q}_+$, then there exist x', y' such that $x = x'$ and $y = y'$ and $x * y = x' * y'$.

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