JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Analytical Ordered Affine Spaces[1]

Henryk Oryszczyszyn
Warsaw University
Białystok

Krzysztof Prażmowski
Warsaw University
Białystok

Summary. In the article with a given arbitrary real linear space we correlate the (ordered) affine space defined in terms of a directed parallelity of segments. The abstract contains a construction of the ordered affine structure associated with a vector space; this is a structure of the type which frequently occurs in geometry and consists of the set of points and a binary relation on segments. For suitable underlying vector spaces we prove that the corresponding affine structures are ordered affine spaces or ordered affine planes, i.e. that they satisfy appropriate axioms. A formal definition of an arbitrary ordered affine space and an arbitrary ordered affine plane is given.

MML Identifier: ANALOAF.

WWW: http://mizar.org/JFM/Vol2/analoaf.html

The articles [5], [2], [4], [3], [7], [6], and [1] provide the notation and terminology for this paper.

We adopt the following rules: V denotes a real linear space, p, q, u, v, w, y denote vectors of V, and a, b denote real numbers.

Let us consider V and let us consider u, v, w, y. The predicate $u,v \upuparrows w,y$ is defined as follows:

(Def. 1) $u = v$ or $w = y$ or there exist a, b such that $0 < a$ and $0 < b$ and $a \cdot (v - u) = b \cdot (y - w)$.

Next we state a number of propositions:

(2)[1] If $0 < a$ and $0 < b$, then $0 < a + b$.

(3) If $a \neq b$, then $0 < a - b$ or $0 < b - a$.

(4) $(w - v) + (v - u) = w - u$.

(6)[2] $w - (u - v) = w + (v - u)$.

(9)[3] If $y + u = v + w$, then $y - w = v - u$.

(10) $a \cdot (u - v) = -a \cdot (v - u)$.

(11) $(a - b) \cdot (u - v) = (b - a) \cdot (v - u)$.

(12) If $a \neq 0$ and $a \cdot u = v$, then $u = a^{-1} \cdot v$.

(13) If $a \neq 0$ and $a \cdot u = v$, then $u = a^{-1} \cdot v$ and if $a \neq 0$ and $u = a^{-1} \cdot v$, then $a \cdot u = v$.

[1] Supported by RPBP.III-24.C6.

[1] The proposition (1) has been removed.
[2] The proposition (5) has been removed.
[3] The propositions (7) and (8) have been removed.

(16)[4] If $u, v \upuparrows w, y$ and $u \neq v$ and $w \neq y$, then there exist a, b such that $a \cdot (v - u) = b \cdot (y - w)$ and $0 < a$ and $0 < b$.

(17) $u, v \upuparrows u, v$.

(18) $u, v \upuparrows w, w$ and $u, u \upuparrows v, w$.

(19) If $u, v \upuparrows v, u$, then $u = v$.

(20) If $p \neq q$ and $p, q \upuparrows u, v$ and $p, q \upuparrows w, y$, then $u, v \upuparrows w, y$.

(21) If $u, v \upuparrows w, y$, then $v, u \upuparrows y, w$ and $w, y \upuparrows u, v$.

(22) If $u, v \upuparrows v, w$, then $u, v \upuparrows u, w$.

(23) If $u, v \upuparrows u, w$, then $u, v \upuparrows v, w$ or $u, w \upuparrows w, v$.

(24) If $v - u = y - w$, then $u, v \upuparrows w, y$.

(25) If $y = (v + w) - u$, then $u, v \upuparrows w, y$ and $u, w \upuparrows v, y$.

(26) If there exist p, q such that $p \neq q$, then for all u, v, w there exists y such that $u, v \upuparrows w, y$ and $u, w \upuparrows v, y$ and $v \neq y$.

(27) If $p \neq v$ and $v, p \upuparrows p, w$, then there exists y such that $u, p \upuparrows p, y$ and $u, v \upuparrows w, y$.

(28) If for all a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$, then $u \neq v$ and $u \neq 0_V$ and $v \neq 0_V$.

(29) If there exist u, v such that for all a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$, then there exist u, v, w, y such that $u, v \not\upuparrows w, y$ and $u, v \not\upuparrows y, w$.

(31)[5] Given p, q such that let given w. Then there exist a, b such that $a \cdot p + b \cdot q = w$. Let given u, v, w, y. Suppose $u, v \not\upuparrows w, y$ and $u, v \not\upuparrows y, w$. Then there exists a vector z of V such that $u, v \upuparrows u, z$ or $u, v \upuparrows z, u$ but $w, y \upuparrows w, z$ or $w, y \upuparrows z, w$.

We introduce affine structures which are extensions of 1-sorted structure and are systems

$\langle$ a carrier, a congruence $\rangle$,

where the carrier is a set and the congruence is a binary relation on $[\!:$the carrier, the carrier$:\!]$.

Let us observe that there exists an affine structure which is non empty and strict.

We adopt the following rules: A_1 denotes a non empty affine structure, a, b, c, d denote elements of A_1, and x, z denote sets.

Let us consider A_1, a, b, c, d. The predicate $a, b \upuparrows c, d$ is defined by:

(Def. 2) $\langle \langle a, b \rangle, \langle c, d \rangle \rangle \in$ the congruence of A_1.

Let us consider V. The functor $\upuparrows_V$ yields a binary relation on $[\!:$the carrier of V, the carrier of $V\,:\!]$ and is defined as follows:

(Def. 3) $\langle x, z \rangle \in \upuparrows_V$ iff there exist u, v, w, y such that $x = \langle u, v \rangle$ and $z = \langle w, y \rangle$ and $u, v \upuparrows w, y$.

One can prove the following proposition

(33)[6] $\langle \langle u, v \rangle, \langle w, y \rangle \rangle \in \upuparrows_V$ iff $u, v \upuparrows w, y$.

Let us consider V. The functor OASpace$V$ yields a strict affine structure and is defined as follows:

(Def. 4) OASpace$V = \langle$the carrier of V, $\upuparrows_V \rangle$.

[4] The propositions (14) and (15) have been removed.
[5] The proposition (30) has been removed.
[6] The proposition (32) has been removed.

Let us consider V. Note that OASpace$V$ is non empty.

Next we state two propositions:

(35)[7] Given u, v such that let a, b be real numbers. If $a \cdot u + b \cdot v = 0_V$, then $a = 0$ and $b = 0$. Then

(i) there exist elements a, b of OASpace$V$ such that $a \neq b$,

(ii) for all elements a, b, c, d, p, q, r, s of OASpace$V$ holds $a,b \upuparrows c,c$ and if $a,b \upuparrows b,a$, then $a = b$ and if $a \neq b$ and $a,b \upuparrows p,q$ and $a,b \upuparrows r,s$, then $p,q \upuparrows r,s$ and if $a,b \upuparrows c,d$, then $b,a \upuparrows d,c$ and if $a,b \upuparrows b,c$, then $a,b \upuparrows a,c$ and if $a,b \upuparrows a,c$, then $a,b \upuparrows b,c$ or $a,c \upuparrows c,b$,

(iii) there exist elements a, b, c, d of OASpace$V$ such that $a,b \not\upuparrows c,d$ and $a,b \not\upuparrows d,c$,

(iv) for all elements a, b, c of OASpace$V$ there exists an element d of OASpace$V$ such that $a,b \upuparrows c,d$ and $a,c \upuparrows b,d$ and $b \neq d$, and

(v) for all elements p, a, b, c of OASpace$V$ such that $p \neq b$ and $b,p \upuparrows p,c$ there exists an element d of OASpace$V$ such that $a,p \upuparrows p,d$ and $a,b \upuparrows c,d$.

(36) Given vectors p, q of V such that let w be a vector of V. Then there exist real numbers a, b such that $a \cdot p + b \cdot q = w$. Let a, b, c, d be elements of OASpace$V$. Suppose $a,b \not\upuparrows c,d$ and $a,b \not\upuparrows d,c$. Then there exists an element t of OASpace$V$ such that $a,b \upuparrows a,t$ or $a,b \upuparrows t,a$ but $c,d \upuparrows c,t$ or $c,d \upuparrows t,c$.

Let I_1 be a non empty affine structure. We say that I_1 is ordered affine space-like if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) For all elements a, b, c, d, p, q, r, s of I_1 holds $a,b \upuparrows c,c$ and if $a,b \upuparrows b,a$, then $a = b$ and if $a \neq b$ and $a,b \upuparrows p,q$ and $a,b \upuparrows r,s$, then $p,q \upuparrows r,s$ and if $a,b \upuparrows c,d$, then $b,a \upuparrows d,c$ and if $a,b \upuparrows b,c$, then $a,b \upuparrows a,c$ and if $a,b \upuparrows a,c$, then $a,b \upuparrows b,c$ or $a,c \upuparrows c,b$,

(ii) there exist elements a, b, c, d of I_1 such that $a,b \not\upuparrows c,d$ and $a,b \not\upuparrows d,c$,

(iii) for all elements a, b, c of I_1 there exists an element d of I_1 such that $a,b \upuparrows c,d$ and $a,c \upuparrows b,d$ and $b \neq d$, and

(iv) for all elements p, a, b, c of I_1 such that $p \neq b$ and $b,p \upuparrows p,c$ there exists an element d of I_1 such that $a,p \upuparrows p,d$ and $a,b \upuparrows c,d$.

Let us note that there exists a non empty affine structure which is strict, non trivial, and ordered affine space-like.

An ordered affine space is a non trivial ordered affine space-like non empty affine structure.

We now state two propositions:

(37) There exist elements a, b of A_1 such that $a \neq b$ and for all elements a, b, c, d, p, q, r, s of A_1 holds $a,b \upuparrows c,c$ and if $a,b \upuparrows b,a$, then $a = b$ and if $a \neq b$ and $a,b \upuparrows p,q$ and $a,b \upuparrows r,s$, then $p,q \upuparrows r,s$ and if $a,b \upuparrows c,d$, then $b,a \upuparrows d,c$ and if $a,b \upuparrows b,c$, then $a,b \upuparrows a,c$ and if $a,b \upuparrows a,c$, then $a,b \upuparrows b,c$ or $a,c \upuparrows c,b$ and there exist elements a, b, c, d of A_1 such that $a,b \not\upuparrows c,d$ and $a,b \not\upuparrows d,c$ and for all elements a, b, c of A_1 there exists an element d of A_1 such that $a,b \upuparrows c,d$ and $a,c \upuparrows b,d$ and $b \neq d$ and for all elements p, a, b, c of A_1 such that $p \neq b$ and $b,p \upuparrows p,c$ there exists an element d of A_1 such that $a,p \upuparrows p,d$ and $a,b \upuparrows c,d$ if and only if A_1 is an ordered affine space.

(38) Given u, v such that let a, b be real numbers. If $a \cdot u + b \cdot v = 0_V$, then $a = 0$ and $b = 0$. Then OASpace$V$ is an ordered affine space.

Let I_1 be an ordered affine space. We say that I_1 is 2-dimensional if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let a, b, c, d be elements of I_1. Suppose $a,b \not\upuparrows c,d$ and $a,b \not\upuparrows d,c$. Then there exists an element p of I_1 such that $a,b \upuparrows a,p$ or $a,b \upuparrows p,a$ but $c,d \upuparrows c,p$ or $c,d \upuparrows p,c$.

[7] The proposition (34) has been removed.

Let us observe that there exists an ordered affine space which is strict and 2-dimensional.
An ordered affine plane is a 2-dimensional ordered affine space.
One can prove the following two propositions:

(50)[8] There exist elements a, b of A_1 such that $a \neq b$ and for all elements a, b, c, d, p, q, r, s of A_1 holds $a,b \upuparrows c,c$ and if $a,b \upuparrows b,a$, then $a = b$ and if $a \neq b$ and $a,b \upuparrows p,q$ and $a,b \upuparrows r,s$, then $p,q \upuparrows r,s$ and if $a,b \upuparrows c,d$, then $b,a \upuparrows d,c$ and if $a,b \upuparrows b,c$, then $a,b \upuparrows a,c$ and if $a,b \upuparrows a,c$, then $a,b \upuparrows b,c$ or $a,c \upuparrows c,b$ and there exist elements a, b, c, d of A_1 such that $a,b \not\upuparrows c,d$ and $a,b \not\upuparrows d,c$ and for all elements a, b, c of A_1 there exists an element d of A_1 such that $a,b \upuparrows c,d$ and $a,c \upuparrows b,d$ and $b \neq d$ and for all elements p, a, b, c of A_1 such that $p \neq b$ and $b,p \upuparrows p,c$ there exists an element d of A_1 such that $a,p \upuparrows p,d$ and $a,b \upuparrows c,d$ and for all elements a, b, c, d of A_1 such that $a,b \not\upuparrows c,d$ and $a,b \not\upuparrows d,c$ there exists an element p of A_1 such that $a,b \upuparrows a,p$ or $a,b \upuparrows p,a$ but $c,d \upuparrows c,p$ or $c,d \upuparrows p,c$ if and only if A_1 is an ordered affine plane.

(51) Given u, v such that

(i) for all real numbers a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$, and

(ii) for every w there exist real numbers a, b such that $w = a \cdot u + b \cdot v$.

Then OASpace$V$ is an ordered affine plane.

References

[1] Józef Białas. Group and field definitions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/realset1.html`.

[2] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[3] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[4] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/domain_1.html`.

[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[6] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/rlvect_1.html`.

[7] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

Received April 11, 1990

Published January 2, 2004

[8] The propositions (39)–(49) have been removed.