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Analytical Metric Affine Spaces and Planes[1]

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Summary. We introduce relations of orthogonality of vectors and of orthogonality of segments (considered as pairs of vectors) in real linear space of dimension two. This enables us to show an example of (in fact anisotropic and satisfying theorem on three perpendiculars) metric affine space (and plane as well). These two types of objects are defined formally as "Mizar" modes. They are to be understood as structures consisting of a point universe and two binary relations on segments — a parallelity relation and orthogonality relation, satisfying appropriate axioms. With every such structure we correlate a structure obtained as a reduct of the given one to the parallelity relation only. Some relationships between metric affine spaces and their affine parts are proved; they enable us to use "affine" facts and constructions in investigating metric affine geometry. We define the notions of line, parallelity of lines and two derived relations of orthogonality: between segments and lines, and between lines. Some basic properties of the introduced notions are proved.

MML Identifier: ANALMETR.

WWW: http://mizar.org/JFM/Vol2/analmetr.html

The articles [8], [2], [10], [7], [3], [5], [11], [9], [6], [4], and [1] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: V denotes a real linear space, u, u_1, u_2, v, v_1, v_2, w, y denote vectors of V, a, a_1, a_2, b, b_1, b_2 denote real numbers, and x, z denote sets.

Let us consider V and let us consider w, y. We say that w, y span the space if and only if:

(Def. 1) For every u there exist a_1, a_2 such that $u = a_1 \cdot w + a_2 \cdot y$ and for all a_1, a_2 such that $a_1 \cdot w + a_2 \cdot y = 0_V$ holds $a_1 = 0$ and $a_2 = 0$.

Let us consider V and let us consider u, v, w, y. We say that u, v are orthogonal w.r.t. w, y if and only if:

(Def. 2) There exist a_1, a_2, b_1, b_2 such that $u = a_1 \cdot w + a_2 \cdot y$ and $v = b_1 \cdot w + b_2 \cdot y$ and $a_1 \cdot b_1 + a_2 \cdot b_2 = 0$.

The following propositions are true:

(5)[1] Let given w, y. Suppose w, y span the space. Then u, v are orthogonal w.r.t. w, y if and only if for all a_1, a_2, b_1, b_2 such that $u = a_1 \cdot w + a_2 \cdot y$ and $v = b_1 \cdot w + b_2 \cdot y$ holds $a_1 \cdot b_1 + a_2 \cdot b_2 = 0$.

(6) w, y are orthogonal w.r.t. w, y.

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[1] The propositions (1)–(4) have been removed.

(7) There exists V and there exist w, y such that w, y span the space.

(8) If u, v are orthogonal w.r.t. w, y, then v, u are orthogonal w.r.t. w, y.

(9) Suppose w, y span the space. Let given u, v. Then u, 0_V are orthogonal w.r.t. w, y and 0_V, v are orthogonal w.r.t. w, y.

(10) If u, v are orthogonal w.r.t. w, y, then $a \cdot u$, $b \cdot v$ are orthogonal w.r.t. w, y.

(11) Suppose u, v are orthogonal w.r.t. w, y. Then $a \cdot u$, v are orthogonal w.r.t. w, y and u, $b \cdot v$ are orthogonal w.r.t. w, y.

(12) If w, y span the space, then for every u there exists v such that u, v are orthogonal w.r.t. w, y and $v \neq 0_V$.

(13) Suppose that

(i) w, y span the space,

(ii) v, u_1 are orthogonal w.r.t. w, y,

(iii) v, u_2 are orthogonal w.r.t. w, y, and

(iv) $v \neq 0_V$.

Then there exist a, b such that $a \cdot u_1 = b \cdot u_2$ but $a \neq 0$ or $b \neq 0$.

(14) Suppose w, y span the space and u, v_1 are orthogonal w.r.t. w, y and u, v_2 are orthogonal w.r.t. w, y. Then u, $v_1 + v_2$ are orthogonal w.r.t. w, y and u, $v_1 - v_2$ are orthogonal w.r.t. w, y.

(15) If w, y span the space and u, u are orthogonal w.r.t. w, y, then $u = 0_V$.

(16) Suppose w, y span the space and u, $u_1 - u_2$ are orthogonal w.r.t. w, y and u_1, $u_2 - u$ are orthogonal w.r.t. w, y. Then u_2, $u - u_1$ are orthogonal w.r.t. w, y.

(17) If w, y span the space and $u \neq 0_V$, then there exists a such that $v - a \cdot u$, u are orthogonal w.r.t. w, y.

(18) $u, v \upuparrows u_1, v_1$ or $u, v \upuparrows v_1, u_1$ iff there exist a, b such that $a \cdot (v - u) = b \cdot (v_1 - u_1)$ but $a \neq 0$ or $b \neq 0$.

(19) $\langle\langle u, v\rangle, \langle u_1, v_1\rangle\rangle \in \lambda(\upuparrows_V)$ iff there exist a, b such that $a \cdot (v - u) = b \cdot (v_1 - u_1)$ but $a \neq 0$ or $b \neq 0$.

Let us consider V and let us consider u, u_1, v, v_1, w, y. We say that u, u_1, v and v_1 are orthogonal w.r.t. w, y if and only if:

(Def. 3) $u_1 - u$, $v_1 - v$ are orthogonal w.r.t. w, y.

Let us consider V and let us consider w, y. The orthogonality determined by w, y in V yields a binary relation on [: the carrier of V, the carrier of V :] and is defined by the condition (Def. 4).

(Def. 4) $\langle x, z\rangle \in$ the orthogonality determined by w, y in V if and only if there exist u, u_1, v, v_1 such that $x = \langle u, u_1\rangle$ and $z = \langle v, v_1\rangle$ and u, u_1, v and v_1 are orthogonal w.r.t. w, y.

In the sequel p, p_1, q, q_1 denote elements of $\Lambda(\text{OASpace}\,V)$.

The following propositions are true:

(22)[2] The carrier of $\Lambda(\text{OASpace}\,V) =$ the carrier of V.

(23) The congruence of $\Lambda(\text{OASpace}\,V) = \lambda(\upuparrows_V)$.

(24) If $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$, then $p, q \upuparrows p_1, q_1$ iff there exist a, b such that $a \cdot (v - u) = b \cdot (v_1 - u_1)$ but $a \neq 0$ or $b \neq 0$.

[2] The propositions (20) and (21) have been removed.

We consider metric affine structures as extensions of affine structure as systems
⟨ a carrier, a congruence, an orthogonality ⟩,
where the carrier is a set, the congruence is a binary relation on [:the carrier, the carrier:], and the orthogonality is a binary relation on [:the carrier, the carrier:].

Let us mention that there exists a metric-affine structure which is non empty.

In the sequel P_1 is a non empty metric-affine structure.

Let us consider P_1 and let a, b, c, d be elements of P_1. The predicate $a,b \upuparrows c,d$ is defined as follows:

(Def. 5) $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of P_1.

The predicate $a,b \perp c,d$ is defined by:

(Def. 6) $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the orthogonality of P_1.

Let us consider V, w, y. The functor $\mathbf{AMSp}(V,w,y)$ yielding a strict metric-affine structure is defined as follows:

(Def. 7) $\mathbf{AMSp}(V,w,y) = \langle$the carrier of V, $\lambda(\upuparrows\restriction_V)$, the orthogonality determined by w, y in $V\rangle$.

Let us consider V, w, y. Observe that $\mathbf{AMSp}(V,w,y)$ is non empty.

We now state the proposition

(28)[3](i) The carrier of $\mathbf{AMSp}(V,w,y) =$ the carrier of V,

(ii) the congruence of $\mathbf{AMSp}(V,w,y) = \lambda(\upuparrows\restriction_V)$, and

(iii) the orthogonality of $\mathbf{AMSp}(V,w,y) =$ the orthogonality determined by w, y in V.

Let us consider P_1. The affine reduct of P_1 yielding a strict affine structure is defined by:

(Def. 8) The affine reduct of $P_1 = \langle$the carrier of P_1, the congruence of $P_1\rangle$.

Let us consider P_1. Note that the affine reduct of P_1 is non empty.

One can prove the following proposition

(30)[4] The affine reduct of $\mathbf{AMSp}(V,w,y) = \Lambda(\mathrm{OASpace}\,V)$.

In the sequel p, p_1, p_2, q, q_1, r, r_1, r_2 denote elements of $\mathbf{AMSp}(V,w,y)$.

One can prove the following propositions:

(31) If $p = u$ and $p_1 = u_1$ and $q = v$ and $q_1 = v_1$, then $p,q \perp p_1,q_1$ iff u, v, u_1 and v_1 are orthogonal w.r.t. w, y.

(32) If $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$, then $p,q \upuparrows p_1,q_1$ iff there exist a, b such that $a \cdot (v-u) = b \cdot (v_1 - u_1)$ but $a \neq 0$ or $b \neq 0$.

(33) If $p,q \perp p_1,q_1$, then $p_1,q_1 \perp p,q$.

(34) If $p,q \perp p_1,q_1$, then $p,q \perp q_1,p_1$.

(35) If w, y span the space, then for all p, q, r holds $p,q \perp r,r$.

(36) If $p,p_1 \perp q,q_1$ and $p,p_1 \upuparrows r,r_1$, then $p = p_1$ or $q,q_1 \perp r,r_1$.

(37) If w, y span the space, then for all p, q, r there exists r_1 such that $p,q \perp r,r_1$ and $r \neq r_1$.

(38) If w, y span the space and $p,p_1 \perp q,q_1$ and $p,p_1 \perp r,r_1$, then $p = p_1$ or $q,q_1 \upuparrows r,r_1$.

(39) If w, y span the space and $p,q \perp r,r_1$ and $p,q \perp r,r_2$, then $p,q \perp r_1,r_2$.

(40) If w, y span the space and $p,q \perp p,q$, then $p = q$.

[3] The propositions (25)–(27) have been removed.
[4] The proposition (29) has been removed.

(41) If w, y span the space and $p,q \perp p_1,p_2$ and $p_1,q \perp p_2,p$, then $p_2,q \perp p,p_1$.

(42) If w, y span the space and $p \neq p_1$, then for every q there exists q_1 such that $p,p_1 \upuparrows p,q_1$ and $p,p_1 \perp q_1,q$.

Let I_1 be a non empty metric-affine structure. We say that I_1 is metric affine space-like if and only if the conditions (Def. 9) are satisfied.

(Def. 9)(i) $\langle$the carrier of I_1, the congruence of $I_1\rangle$ is an affine space,

(ii) for all elements a, b, c, d, p, q, r, s of I_1 holds if $a,b \perp a,b$, then $a = b$ and $a,b \perp c,c$ and if $a,b \perp c,d$, then $a,b \perp d,c$ and $c,d \perp a,b$ and if $a,b \perp p,q$ and $a,b \upuparrows r,s$, then $p,q \perp r,s$ or $a = b$ and if $a,b \perp p,q$ and $a,b \perp p,s$, then $a,b \perp q,s$,

(iii) for all elements a, b, c of I_1 such that $a \neq b$ there exists an element x of I_1 such that $a,b \upuparrows a,x$ and $a,b \perp x,c$, and

(iv) for all elements a, b, c of I_1 there exists an element x of I_1 such that $a,b \perp c,x$ and $c \neq x$.

Let us mention that there exists a non empty metric-affine structure which is strict and metric affine space-like.

A metric affine space is a metric affine space-like non empty metric-affine structure.

We now state the proposition

(44)[5] If w, y span the space, then $\mathbf{AMSp}(V,w,y)$ is a metric affine space.

Let I_1 be a non empty metric-affine structure. We say that I_1 is metric affine plane-like if and only if the conditions (Def. 10) are satisfied.

(Def. 10)(i) $\langle$the carrier of I_1, the congruence of $I_1\rangle$ is an affine plane,

(ii) for all elements a, b, c, d, p, q, r, s of I_1 holds if $a,b \perp a,b$, then $a = b$ and $a,b \perp c,c$ and if $a,b \perp c,d$, then $a,b \perp d,c$ and $c,d \perp a,b$ and if $a,b \perp p,q$ and $a,b \upuparrows r,s$, then $p,q \perp r,s$ or $a = b$ and if $a,b \perp p,q$ and $a,b \perp r,s$, then $p,q \upuparrows r,s$ or $a = b$, and

(iii) for all elements a, b, c of I_1 there exists an element x of I_1 such that $a,b \perp c,x$ and $c \neq x$.

Let us observe that there exists a non empty metric-affine structure which is strict and metric affine plane-like.

A metric affine plane is a metric affine plane-like non empty metric-affine structure.

The following three propositions are true:

(46)[6] If w, y span the space, then $\mathbf{AMSp}(V,w,y)$ is a metric affine plane.

(47) For every set x holds x is an element of P_1 iff x is an element of the affine reduct of P_1.

(48) Let a, b, c, d be elements of P_1 and a', b', c', d' be elements of the affine reduct of P_1. If $a = a'$ and $b = b'$ and $c = c'$ and $d = d'$, then $a,b \upuparrows c,d$ iff $a',b' \upuparrows c',d'$.

Let P_1 be a metric affine space. One can verify that the affine reduct of P_1 is affine space-like and non trivial.

Let P_1 be a metric affine plane. Observe that the affine reduct of P_1 is 2-dimensional, affine space-like, and non trivial.

We now state the proposition

(49) Every metric affine plane is a metric affine space.

Let us note that every non empty metric-affine structure which is metric affine plane-like is also metric affine space-like.

One can prove the following two propositions:

[5] The proposition (43) has been removed.

[6] The proposition (45) has been removed.

(50) Let P_1 be a metric affine space. Suppose the affine reduct of P_1 is an affine plane. Then P_1 is a metric affine plane.

(51) Let P_1 be a non empty metric-affine structure. Then P_1 is metric affine plane-like if and only if the following conditions are satisfied:

(i) there exist elements a, b of P_1 such that $a \neq b$, and

(ii) for all elements a, b, c, d, p, q, r, s of P_1 holds $a,b \downharpoonleft\upharpoonright b,a$ and $a,b \downharpoonleft\upharpoonright c,c$ and if $a,b \downharpoonleft\upharpoonright p,q$ and $a,b \downharpoonleft\upharpoonright r,s$, then $p,q \downharpoonleft\upharpoonright r,s$ or $a = b$ and if $a,b \downharpoonleft\upharpoonright a,c$, then $b,a \downharpoonleft\upharpoonright b,c$ and there exists an element x of P_1 such that $a,b \downharpoonleft\upharpoonright c,x$ and $a,c \downharpoonleft\upharpoonright b,x$ and there exist elements x, y, z of P_1 such that $x,y \not\downharpoonleft\upharpoonright x,z$ and there exists an element x of P_1 such that $a,b \downharpoonleft\upharpoonright c,x$ and $c \neq x$ and if $a,b \downharpoonleft\upharpoonright b,d$ and $b \neq a$, then there exists an element x of P_1 such that $c,b \downharpoonleft\upharpoonright b,x$ and $c,a \downharpoonleft\upharpoonright d,x$ and if $a,b \perp a,b$, then $a = b$ and $a,b \perp c,c$ and if $a,b \perp c,d$, then $a,b \perp d,c$ and $c,d \perp a,b$ and if $a,b \perp p,q$ and $a,b \downharpoonleft\upharpoonright r,s$, then $p,q \perp r,s$ or $a = b$ and if $a,b \perp p,q$ and $a,b \perp r,s$, then $p,q \downharpoonleft\upharpoonright r,s$ or $a = b$ and there exists an element x of P_1 such that $a,b \perp c,x$ and $c \neq x$ and if $a,b \not\downharpoonleft\upharpoonright c,d$, then there exists an element x of P_1 such that $a,b \downharpoonleft\upharpoonright a,x$ and $c,d \downharpoonleft\upharpoonright c,x$.

In the sequel a, b, c, d, p, q are elements of P_1.

Let us consider P_1 and let us consider a, b, c. The predicate $\mathbf{L}(a,b,c)$ is defined as follows:

(Def. 11) $a,b \downharpoonleft\upharpoonright a,c$.

Let us consider P_1, a, b. The functor $\mathrm{Line}(a,b)$ yielding a subset of P_1 is defined by:

(Def. 12) For every element x of P_1 holds $x \in \mathrm{Line}(a,b)$ iff $\mathbf{L}(a,b,x)$.

In the sequel A, K, M denote subsets of P_1.

Let us consider P_1 and let us consider A. We say that A is line if and only if:

(Def. 13) There exist a, b such that $a \neq b$ and $A = \mathrm{Line}(a,b)$.

We introduce A is a line as a synonym of A is line.

Next we state four propositions:

(55)[7] Let P_1 be a metric affine space, a, b, c be elements of P_1, and a', b', c' be elements of the affine reduct of P_1. If $a = a'$ and $b = b'$ and $c = c'$, then $\mathbf{L}(a,b,c)$ iff $\mathbf{L}(a',b',c')$.

(56) Let P_1 be a metric affine space, a, b be elements of P_1, and a', b' be elements of the affine reduct of P_1. If $a = a'$ and $b = b'$, then $\mathrm{Line}(a,b) = \mathrm{Line}(a',b')$.

(57) For every set X holds X is a subset of P_1 iff X is a subset of the affine reduct of P_1.

(58) Let P_1 be a metric affine space, X be a subset of P_1, and Y be a subset of the affine reduct of P_1. If $X = Y$, then X is a line iff Y is a line.

Let us consider P_1 and let us consider a, b, K. The predicate $a,b \perp K$ is defined as follows:

(Def. 14) There exist p, q such that $p \neq q$ and $K = \mathrm{Line}(p,q)$ and $a,b \perp p,q$.

Let us consider P_1 and let us consider K, M. The predicate $K \perp M$ is defined as follows:

(Def. 15) There exist p, q such that $p \neq q$ and $K = \mathrm{Line}(p,q)$ and $p,q \perp M$.

Let us consider P_1 and let us consider K, M. We say that K // M if and only if:

(Def. 16) There exist a, b, c, d such that $a \neq b$ and $c \neq d$ and $K = \mathrm{Line}(a,b)$ and $M = \mathrm{Line}(c,d)$ and $a,b \downharpoonleft\upharpoonright c,d$.

Next we state three propositions:

(62)[8] If $a,b \perp K$, then K is a line and if $K \perp M$, then K is a line and M is a line.

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[7] The propositions (52)–(54) have been removed.

[8] The propositions (59)–(61) have been removed.

(63) $K \perp M$ iff there exist a, b, c, d such that $a \neq b$ and $c \neq d$ and $K = \mathrm{Line}(a,b)$ and $M = \mathrm{Line}(c,d)$ and $a,b \perp c,d$.

(64) Let P_1 be a metric affine space, M, N be subsets of P_1, and M', N' be subsets of the affine reduct of P_1. If $M = M'$ and $N = N'$, then $M \mathbin{//} N$ iff $M' \mathbin{//} N'$.

We use the following convention: P_1 denotes a metric affine space, A, K, M, N denote subsets of P_1, and a, b, c, d, p, q, r, s denote elements of P_1.

We now state several propositions:

(65) If K is a line, then $a,a \perp K$.

(66) If $a,b \perp K$ and if $a,b \upuparrows c,d$ or $c,d \upuparrows a,b$ and if $a \neq b$, then $c,d \perp K$.

(67) If $a,b \perp K$, then $b,a \perp K$.

(68) If $K \mathbin{//} M$, then $M \mathbin{//} K$.

(69) If $r,s \perp K$ and if $K \mathbin{//} M$ or $M \mathbin{//} K$, then $r,s \perp M$.

(70) If $K \perp M$, then $M \perp K$.

Let P_1 be a metric affine space and let K, M be subsets of P_1. Let us note that the predicate $K \mathbin{//} M$ is symmetric. Let us note that the predicate $K \perp M$ is symmetric.

One can prove the following propositions:

(71) If $a \in K$ and $b \in K$ and $a,b \perp K$, then $a = b$.

(72) $K \not\perp K$.

(73) If $K \perp M$ or $M \perp K$ and if $K \mathbin{//} N$ or $N \mathbin{//} K$, then $M \perp N$ and $N \perp M$.

(74) If $K \mathbin{//} N$, then $K \not\perp N$.

(75) If $a \in K$ and $b \in K$ and $c,d \perp K$, then $c,d \perp a,b$ and $a,b \perp c,d$.

(76) If $a \in K$ and $b \in K$ and $a \neq b$ and K is a line, then $K = \mathrm{Line}(a,b)$.

(77) If $a \in K$ and $b \in K$ and $a \neq b$ and K is a line and $a,b \perp c,d$ or $c,d \perp a,b$, then $c,d \perp K$.

(78) If $a \in M$ and $b \in M$ and $c \in N$ and $d \in N$ and $M \perp N$, then $a,b \perp c,d$.

(79) If $p \in M$ and $p \in N$ and $a \in M$ and $b \in N$ and $a \neq b$ and $a \in K$ and $b \in K$ and $A \perp M$ and $A \perp N$ and K is a line, then $A \perp K$.

(80) $b,c \perp a,a$ and $a,a \perp b,c$ and $b,c \upuparrows a,a$ and $a,a \upuparrows b,c$.

(81) If $a,b \upuparrows c,d$, then $a,b \upuparrows d,c$ and $b,a \upuparrows c,d$ and $b,a \upuparrows d,c$ and $c,d \upuparrows a,b$ and $c,d \upuparrows b,a$ and $d,c \upuparrows a,b$ and $d,c \upuparrows b,a$.

(82) If $p \neq q$ and if $p,q \upuparrows a,b$ and $p,q \upuparrows c,d$ or $p,q \upuparrows a,b$ and $c,d \upuparrows p,q$ or $a,b \upuparrows p,q$ and $c,d \upuparrows p,q$ or $a,b \upuparrows p,q$ and $p,q \upuparrows c,d$, then $a,b \upuparrows c,d$.

(83) If $a,b \perp c,d$, then $a,b \perp d,c$ and $b,a \perp c,d$ and $b,a \perp d,c$ and $c,d \perp a,b$ and $c,d \perp b,a$ and $d,c \perp a,b$ and $d,c \perp b,a$.

(84) Suppose that

(i) $p \neq q$, and

(ii) $p,q \upuparrows a,b$ and $p,q \perp c,d$ or $p,q \upuparrows c,d$ and $p,q \perp a,b$ or $p,q \upuparrows a,b$ and $c,d \perp p,q$ or $p,q \upuparrows c,d$ and $a,b \perp p,q$ or $a,b \upuparrows p,q$ and $c,d \perp p,q$ or $c,d \upuparrows p,q$ and $a,b \perp p,q$ or $a,b \upuparrows p,q$ and $p,q \perp c,d$ or $c,d \upuparrows p,q$ and $p,q \perp a,b$.

Then $a,b \perp c,d$.

We use the following convention: P_1 is a metric affine plane, K, M, N are subsets of P_1, and x, a, b, c, d, p, q are elements of P_1.

We now state several propositions:

(85) If $p \neq q$ and if $p,q \perp a,b$ and $p,q \perp c,d$ or $p,q \perp a,b$ and $c,d \perp p,q$ or $a,b \perp p,q$ and $c,d \perp p,q$ or $a,b \perp p,q$ and $p,q \perp c,d$, then $a,b \downharpoonleft\!\upharpoonright c,d$.

(86) If $a \in M$ and $b \in M$ and $a \neq b$ and M is a line and $c \in N$ and $d \in N$ and $c \neq d$ and N is a line and $a,b \downharpoonleft\!\upharpoonright c,d$, then $M \mathbin{/\!/} N$.

(87) If $K \perp M$ or $M \perp K$ and if $K \perp N$ or $N \perp K$, then $M \mathbin{/\!/} N$ and $N \mathbin{/\!/} M$.

(88) If $M \perp N$, then there exists p such that $p \in M$ and $p \in N$.

(89) If $a,b \perp c,d$, then there exists p such that $\mathbf{L}(a,b,p)$ and $\mathbf{L}(c,d,p)$.

(90) If $a,b \perp K$, then there exists p such that $\mathbf{L}(a,b,p)$ and $p \in K$.

(91) There exists x such that $a,x \perp p,q$ and $\mathbf{L}(p,q,x)$.

(92) If K is a line, then there exists x such that $a,x \perp K$ and $x \in K$.

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