JOURNAL OF FORMALIZED MATHEMATICS
Volume 8, Released 1996, Published 2003
Inst. of Computer Science, Univ. of Białystok

Examples of Category Structures

Andrzej Trybulec
Warsaw University
Białystok

Summary. We continue the formalization of the category theory.

MML Identifier: `ALTCAT_2`.

WWW: `http://mizar.org/JFM/Vol8/altcat_2.html`

The articles [14], [6], [19], [20], [15], [3], [4], [2], [13], [1], [8], [7], [9], [5], [16], [12], [18], [17], [10], and [11] provide the notation and terminology for this paper.

1. PRELIMINARIES

The following proposition is true

(1) For all sets X_1, X_2 and for all sets a_1, a_2 holds $[: X_1 \longmapsto a_1, X_2 \longmapsto a_2 :] = [: X_1, X_2 :] \longmapsto \langle a_1, a_2 \rangle$.

Let I be a set. One can check that $\mathbf{0}_I$ is function yielding.

Next we state two propositions:

(2) For all functions f, g holds $\curvearrowleft(g \cdot f) = g \cdot \curvearrowleft f$.

(3) For all functions f, g, h holds $\curvearrowleft(f \cdot [: g, h :]) = \curvearrowleft f \cdot [: h, g :]$.

Let f be a function yielding function. Observe that $\curvearrowleft f$ is function yielding.

Next we state the proposition

(4) Let I be a set and A, B, C be many sorted sets indexed by I. Suppose A is transformable to B. Let F be a many sorted function from A into B and G be a many sorted function from B into C. Then $G \circ F$ is a many sorted function from A into C.

Let I be a set and let A be a many sorted set indexed by $[: I, I :]$. Then $\curvearrowleft A$ is a many sorted set indexed by $[: I, I :]$.

Next we state the proposition

(5) Let I_1 be a set, I_2 be a non empty set, f be a function from I_1 into I_2, B, C be many sorted sets indexed by I_2, and G be a many sorted function from B into C. Then $G \cdot f$ is a many sorted function from $B \cdot f$ into $C \cdot f$.

Let I be a set, let A, B be many sorted sets indexed by $[: I, I :]$, and let F be a many sorted function from A into B. Then $\curvearrowleft F$ is a many sorted function from $\curvearrowleft A$ into $\curvearrowleft B$.

Next we state the proposition

(6) Let I_1, I_2 be non empty sets, M be a many sorted set indexed by $[: I_1, I_2 :]$, o_1 be an element of I_1, and o_2 be an element of I_2. Then $(\curvearrowleft M)(o_2, o_1) = M(o_1, o_2)$.

Let I_1 be a set and let f, g be many sorted functions indexed by I_1. Then $g \circ f$ is a many sorted function indexed by I_1.

2. An auxiliary notion

Let f, g be functions. The predicate $f \dot{\subseteq} g$ is defined as follows:

(Def. 1) $\operatorname{dom} f \subseteq \operatorname{dom} g$ and for every set i such that $i \in \operatorname{dom} f$ holds $f(i) \subseteq g(i)$.

Let us note that the predicate $f \dot{\subseteq} g$ is reflexive.

Let I, J be sets, let A be a many sorted set indexed by I, and let B be a many sorted set indexed by J. Let us observe that $A \dot{\subseteq} B$ if and only if:

(Def. 2) $I \subseteq J$ and for every set i such that $i \in I$ holds $A(i) \subseteq B(i)$.

Next we state three propositions:

(8)[1] Let I, J be sets, A be a many sorted set indexed by I, and B be a many sorted set indexed by J. If $A \dot{\subseteq} B$ and $B \dot{\subseteq} A$, then $A = B$.

(9) Let I, J, K be sets, A be a many sorted set indexed by I, B be a many sorted set indexed by J, and C be a many sorted set indexed by K. If $A \dot{\subseteq} B$ and $B \dot{\subseteq} C$, then $A \dot{\subseteq} C$.

(10) Let I be a set, A be a many sorted set indexed by I, and B be a many sorted set indexed by I. Then $A \dot{\subseteq} B$ if and only if $A \subseteq B$.

3. A bit of lambda calculus

In this article we present several logical schemes. The scheme *OnSingletons* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\langle o, \mathcal{F}(o)\rangle; o$ ranges over elements of $\mathcal{A} : \mathcal{P}[o]\}$ is a function

for all values of the parameters.

The scheme *DomOnSingletons* deals with a non empty set $\mathcal{A}$, a function $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\operatorname{dom} \mathcal{B} = \{o; o$ ranges over elements of $\mathcal{A} : \mathcal{P}[o]\}$

provided the following condition is met:

- $\mathcal{B} = \{\langle o, \mathcal{F}(o)\rangle; o$ ranges over elements of $\mathcal{A} : \mathcal{P}[o]\}$.

The scheme *ValOnSingletons* deals with a non empty set $\mathcal{A}$, a function $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{B}(\mathcal{C}) = \mathcal{F}(\mathcal{C})$

provided the parameters have the following properties:

- $\mathcal{B} = \{\langle o, \mathcal{F}(o)\rangle; o$ ranges over elements of $\mathcal{A} : \mathcal{P}[o]\}$, and
- $\mathcal{P}[\mathcal{C}]$.

4. More on old categories

The following propositions are true:

(11) For every category C and for all objects i, j, k of C holds $[: \hom(j,k), \hom(i,j) :] \subseteq \operatorname{dom}$(the composition of C).

(12) For every category C and for all objects i, j, k of C holds (the composition of $C)^\circ [: \hom(j,k), \hom(i,j) :] \subseteq \hom(i,k)$.

[1] The proposition (7) has been removed.

Let C be a category structure. The functor $\mathrm{HomSets}_C$ yielding a many sorted set indexed by $[:$the objects of C, the objects of $C:]$ is defined as follows:

(Def. 3) For all objects i, j of C holds $\mathrm{HomSets}_C(i, j) = \hom(i, j)$.

We now state the proposition

(13) For every category C and for every object i of C holds $\mathrm{id}_i \in \mathrm{HomSets}_C(i, i)$.

Let C be a category. The functor $\mathrm{Composition}_C$ yields a binary composition of $\mathrm{HomSets}_C$ and is defined as follows:

(Def. 4) For all objects i, j, k of C holds $\mathrm{Composition}_C(i, j, k) =$ (the composition of $C) \restriction [:\mathrm{HomSets}_C(j, k), \mathrm{HomSets}_C(i, j):]$.

The following propositions are true:

(14) Let C be a category and i, j, k be objects of C. Suppose $\hom(i, j) \neq \emptyset$ and $\hom(j, k) \neq \emptyset$. Let f be a morphism from i to j and g be a morphism from j to k. Then $\mathrm{Composition}_C(i, j, k)(g, f) = g \cdot f$.

(15) For every category C holds $\mathrm{Composition}_C$ is associative.

(16) For every category C holds $\mathrm{Composition}_C$ has left units and right units.

5. Transforming an Old Category into a New One

Let C be a category. The functor $\mathrm{Alter}(C)$ yields a strict non empty category structure and is defined by:

(Def. 5) $\mathrm{Alter}(C) = \langle$the objects of $C, \mathrm{HomSets}_C, \mathrm{Composition}_C\rangle$.

The following three propositions are true:

(17) For every category C holds $\mathrm{Alter}(C)$ is associative.

(18) For every category C holds $\mathrm{Alter}(C)$ has units.

(19) For every category C holds $\mathrm{Alter}(C)$ is transitive.

Let C be a category. One can verify that $\mathrm{Alter}(C)$ is transitive and associative and has units.

6. More on New Categories

Let us observe that there exists a graph which is non empty and strict.

Let C be a graph. We say that C is reflexive if and only if:

(Def. 6) For every set x such that $x \in$ the carrier of C holds (the arrows of $C)(x, x) \neq \emptyset$.

Let C be a non empty graph. Let us observe that C is reflexive if and only if:

(Def. 7) For every object o of C holds $\langle o, o\rangle \neq \emptyset$.

Let C be a non empty transitive category structure. Let us observe that C is associative if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let o_1, o_2, o_3, o_4 be objects of C, f be a morphism from o_1 to o_2, g be a morphism from o_2 to o_3, and h be a morphism from o_3 to o_4. If $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_3\rangle \neq \emptyset$ and $\langle o_3, o_4\rangle \neq \emptyset$, then $(h \cdot g) \cdot f = h \cdot (g \cdot f)$.

Let C be a non empty category structure. Let us observe that C has units if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let o be an object of C. Then

(i) $\langle o, o\rangle \neq \emptyset$, and

(ii) there exists a morphism i from o to o such that for every object o' of C and for every morphism m' from o' to o and for every morphism m'' from o to o' holds if $\langle o', o\rangle \neq \emptyset$, then $i \cdot m' = m'$ and if $\langle o, o'\rangle \neq \emptyset$, then $m'' \cdot i = m''$.

Let us note that every non empty category structure which has units is also reflexive.
One can verify that there exists a graph which is non empty and reflexive.
Let us observe that there exists a category structure which is non empty and reflexive.

7. The Empty Category

The strict category structure $\emptyset_{CAT}$ is defined by:

(Def. 10) The carrier of $\emptyset_{CAT}$ is empty.

One can verify that $\emptyset_{CAT}$ is empty.
Let us observe that there exists a category structure which is empty and strict.
We now state the proposition

(20) For every empty strict category structure E holds $E = \emptyset_{CAT}$.

8. Subcategories

Let C be a category structure. A category structure is said to be a substructure of C if it satisfies the conditions (Def. 11).

(Def. 11)(i) The carrier of it $\subseteq$ the carrier of C,

(ii) the arrows of it $\dot{\subseteq}$ the arrows of C, and

(iii) the composition of it $\dot{\subseteq}$ the composition of C.

In the sequel C, C_1, C_2, C_3 denote category structures.
One can prove the following propositions:

(21) C is a substructure of C.

(22) If C_1 is a substructure of C_2 and C_2 is a substructure of C_3, then C_1 is a substructure of C_3.

(23) Let C_1, C_2 be category structures. Suppose C_1 is a substructure of C_2 and C_2 is a substructure of C_1. Then the category structure of $C_1 =$ the category structure of C_2.

Let C be a category structure. Note that there exists a substructure of C which is strict.

Let C be a non empty category structure and let o be an object of C. The functor $\Box\restriction o$ yields a strict substructure of C and is defined by the conditions (Def. 12).

(Def. 12)(i) The carrier of $\Box\restriction o = \{o\}$,

(ii) the arrows of $\Box\restriction o = [\langle o, o\rangle \mapsto \langle o, o\rangle]$, and

(iii) the composition of $\Box\restriction o = \langle o, o, o\rangle \dot{\longmapsto}$(the composition of $C)(o, o, o)$.

In the sequel C is a non empty category structure and o is an object of C.
Next we state the proposition

(24) For every object o' of $\Box\restriction o$ holds $o' = o$.

Let C be a non empty category structure and let o be an object of C. One can check that $\Box\restriction o$ is transitive and non empty.

Let C be a non empty category structure. Observe that there exists a substructure of C which is transitive, non empty, and strict.

The following proposition is true

(25) Let C be a transitive non empty category structure and D_1, D_2 be transitive non empty substructures of C. Suppose the carrier of $D_1 \subseteq$ the carrier of D_2 and the arrows of $D_1 \dot{\subseteq}$ the arrows of D_2. Then D_1 is a substructure of D_2.

Let C be a category structure and let D be a substructure of C. We say that D is full if and only if:

(Def. 13) The arrows of $D =$ (the arrows of C)$\upharpoonright$[: the carrier of D, the carrier of D :].

Let C be a non empty category structure with units and let D be a substructure of C. We say that D is id-inheriting if and only if:

(Def. 14)(i) For every object o of D and for every object o' of C such that $o = o'$ holds $\mathrm{id}_{o'} \in \langle o, o \rangle$ if D is non empty,

(ii) TRUE, otherwise.

Let C be a category structure. Note that there exists a substructure of C which is full and strict.

Let C be a non empty category structure. Observe that there exists a substructure of C which is full, non empty, and strict.

Let C be a category and let o be an object of C. Observe that $\square\upharpoonright o$ is full and id-inheriting.

Let C be a category. Observe that there exists a substructure of C which is full, id-inheriting, non empty, and strict.

In the sequel C is a non empty transitive category structure.

The following propositions are true:

(26) Let D be a substructure of C. Suppose the carrier of $D =$ the carrier of C and the arrows of $D =$ the arrows of C. Then the category structure of $D =$ the category structure of C.

(27) Let D_1, D_2 be non empty transitive substructures of C. Suppose the carrier of $D_1 =$ the carrier of D_2 and the arrows of $D_1 =$ the arrows of D_2. Then the category structure of $D_1 =$ the category structure of D_2.

(28) Let D be a full substructure of C. Suppose the carrier of $D =$ the carrier of C. Then the category structure of $D =$ the category structure of C.

(29) Let C be a non empty category structure, D be a full non empty substructure of C, o_1, o_2 be objects of C, and p_1, p_2 be objects of D. If $o_1 = p_1$ and $o_2 = p_2$, then $\langle o_1, o_2 \rangle = \langle p_1, p_2 \rangle$.

(30) For every non empty category structure C and for every non empty substructure D of C holds every object of D is an object of C.

Let C be a transitive non empty category structure. Observe that every substructure of C which is full and non empty is also transitive.

Next we state three propositions:

(31) Let D_1, D_2 be full non empty substructures of C. Suppose the carrier of $D_1 =$ the carrier of D_2. Then the category structure of $D_1 =$ the category structure of D_2.

(32) Let C be a non empty category structure, D be a non empty substructure of C, o_1, o_2 be objects of C, and p_1, p_2 be objects of D. If $o_1 = p_1$ and $o_2 = p_2$, then $\langle p_1, p_2 \rangle \subseteq \langle o_1, o_2 \rangle$.

(33) Let C be a non empty transitive category structure, D be a non empty transitive substructure of C, and p_1, p_2, p_3 be objects of D. Suppose $\langle p_1, p_2 \rangle \neq \emptyset$ and $\langle p_2, p_3 \rangle \neq \emptyset$. Let o_1, o_2, o_3 be objects of C. Suppose $o_1 = p_1$ and $o_2 = p_2$ and $o_3 = p_3$. Let f be a morphism from o_1 to o_2, g be a morphism from o_2 to o_3, f_1 be a morphism from p_1 to p_2, and g_1 be a morphism from p_2 to p_3. If $f = f_1$ and $g = g_1$, then $g \cdot f = g_1 \cdot f_1$.

Let C be an associative transitive non empty category structure. Observe that every non empty substructure of C which is transitive is also associative.

One can prove the following proposition

(34) Let C be a non empty category structure, D be a non empty substructure of C, o_1, o_2 be objects of C, and p_1, p_2 be objects of D. If $o_1 = p_1$ and $o_2 = p_2$ and $\langle p_1, p_2 \rangle \neq \emptyset$, then every morphism from p_1 to p_2 is a morphism from o_1 to o_2.

Let C be a transitive non empty category structure with units. One can verify that every non empty substructure of C which is id-inheriting and transitive has also units.

Let C be a category. Observe that there exists a non empty substructure of C which is id-inheriting and transitive.

Let C be a category. A subcategory of C is an id-inheriting transitive substructure of C.

The following proposition is true

(35) Let C be a category, D be a non empty subcategory of C, o be an object of D, and o' be an object of C. If $o = o'$, then $\mathrm{id}_o = \mathrm{id}_{o'}$.

REFERENCES

[1] Czesław Byliński. Basic functions and operations on functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_3.html.

[2] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[5] Czesław Byliński. Introduction to categories and functors. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/cat_1.html.

[6] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[7] Czesław Byliński. A classical first order language. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/cqc_lang.html.

[8] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/funct_4.html.

[9] Czesław Byliński. Cartesian categories. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/cat_4.html.

[10] Artur Korniłowicz. On the group of automorphisms of universal algebra and many sorted algebra. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/autalg_1.html.

[11] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_3.html.

[12] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pralg_1.html.

[13] Michał Muzalewski and Wojciech Skaba. Three-argument operations and four-argument operations. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/multop_1.html.

[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[15] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/mcart_1.html.

[16] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[17] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_1.html.

[18] Andrzej Trybulec. Categories without uniqueness of cod and dom. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/altcat_1.html.

[19] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[20] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received January 22, 1996

Published January 2, 2004