Technical Preliminaries to Algebraic Specifications

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The articles [14], [18], [19], [6], [17], [5], [20], [8], [9], [10], [15], [11], [7], [1], [16], [13], [2], [12], [4], and [3] provide the notation and terminology for this paper.

1. PRELIMINARIES

The following propositions are true:

(1) For all functions f, g, h such that $\operatorname{dom} f \cap \operatorname{dom} g \subseteq \operatorname{dom} h$ holds $f + \cdot g + \cdot h = g + \cdot f + \cdot h$.

(2) For all functions f, g, h such that $f \subseteq g$ and $\operatorname{rng} h \cap \operatorname{dom} g \subseteq \operatorname{dom} f$ holds $g \cdot h = f \cdot h$.

(3) For all functions f, g, h such that $\operatorname{dom} f \subseteq \operatorname{rng} g$ and $\operatorname{dom} f$ misses $\operatorname{rng} h$ and $g^{\circ} \operatorname{dom} h$ misses $\operatorname{dom} f$ holds $f \cdot (g + \cdot h) = f \cdot g$.

(4) For all functions f_1, f_2, g_1, g_2 such that $f_1 \approx f_2$ and $g_1 \approx g_2$ holds $f_1 \cdot g_1 \approx f_2 \cdot g_2$.

(5) Let X_1, Y_1, X_2, Y_2 be non empty sets, f be a function from X_1 into X_2, and g be a function from Y_1 into Y_2. If $f \subseteq g$, then $f^* \subseteq g^*$.

(6) Let X_1, Y_1, X_2, Y_2 be non empty sets, f be a function from X_1 into X_2, and g be a function from Y_1 into Y_2. If $f \approx g$, then $f^* \approx g^*$.

Let X be a set and let f be a function. The functor X-indexing f yields a many sorted set indexed by X and is defined by:

(Def. 1) X-indexing $f = \operatorname{id}_X + \cdot f \upharpoonright X$.

One can prove the following propositions:

(7) For every set X and for every function f holds $\operatorname{rng}(X\text{-indexing } f) = (X \setminus \operatorname{dom} f) \cup f^{\circ} X$.

(8) For every non empty set X and for every function f and for every element x of X holds $(X\text{-indexing } f)(x) = (\operatorname{id}_X + \cdot f)(x)$.

(9) For all sets X, x and for every function f such that $x \in X$ holds if $x \in \operatorname{dom} f$, then $(X\text{-indexing } f)(x) = f(x)$ and if $x \notin \operatorname{dom} f$, then $(X\text{-indexing } f)(x) = x$.

(10) For every set X and for every function f such that $\operatorname{dom} f = X$ holds X-indexing $f = f$.

(11) For every set X and for every function f holds X-indexing$(X\text{-indexing } f) = X$-indexing f.

(12) For every set X and for every function f holds X-indexing$(\operatorname{id}_X + \cdot f) = X$-indexing f.

(13)   For every set X and for every function f such that $f \subseteq \mathrm{id}_X$ holds X-indexing $f = \mathrm{id}_X$.

(14)   For every set X holds X-indexing $\emptyset = \mathrm{id}_X$.

(15)   For every set X and for every function f holds X-indexing $f{\restriction}X = X$-indexing f.

(16)   For every set X and for every function f such that $X \subseteq \mathrm{dom}\, f$ holds X-indexing $f = f{\restriction}X$.

(17)   For every function f holds $\emptyset$-indexing $f = \emptyset$.

(18)   For all sets X, Y and for every function f such that $X \subseteq Y$ holds $(Y$-indexing $f){\restriction}X = X$-indexing f.

(19)   For all sets X, Y and for every function f holds $(X \cup Y)$-indexing $f = (X$-indexing $f) + \cdot (Y$-indexing $f)$.

(20)   For all sets X, Y and for every function f holds X-indexing $f \approx Y$-indexing f.

(21)   For all sets X, Y and for every function f holds $(X \cup Y)$-indexing $f = (X$-indexing $f) \cup (Y$-indexing $f)$.

(22)   For every non empty set X and for all functions f, g such that $\mathrm{rng}\, g \subseteq X$ holds $(X$-indexing $f) \cdot g = (\mathrm{id}_X + \cdot f) \cdot g$.

(23)   For all functions f, g such that $\mathrm{dom}\, f$ misses $\mathrm{dom}\, g$ and $\mathrm{rng}\, g$ misses $\mathrm{dom}\, f$ and for every set X holds $f \cdot (X$-indexing $g) = f{\restriction}X$.

Let f be a function. A function is called a rng-retraction of f if:

(Def. 2) $\mathrm{dom}\,\mathrm{it} = \mathrm{rng}\, f$ and $f \cdot \mathrm{it} = \mathrm{id}_{\mathrm{rng}\, f}$.

One can prove the following propositions:

(24)   For every function f and for every rng-retraction g of f holds $\mathrm{rng}\, g \subseteq \mathrm{dom}\, f$.

(25)   Let f be a function, g be a rng-retraction of f, and x be a set. If $x \in \mathrm{rng}\, f$, then $g(x) \in \mathrm{dom}\, f$ and $f(g(x)) = x$.

(26)   For every function f such that f is one-to-one holds f^{-1} is a rng-retraction of f.

(27)   For every function f such that f is one-to-one and for every rng-retraction g of f holds $g = f^{-1}$.

(28)   Let f_1, f_2 be functions. Suppose $f_1 \approx f_2$. Let g_1 be a rng-retraction of f_1 and g_2 be a rng-retraction of f_2. Then $g_1 + \cdot g_2$ is a rng-retraction of $f_1 + \cdot f_2$.

(29)   Let f_1, f_2 be functions. Suppose $f_1 \subseteq f_2$. Let g_1 be a rng-retraction of f_1. Then there exists a rng-retraction g_2 of f_2 such that $g_1 \subseteq g_2$.

2.   REPLACEMENT IN SIGNATURE

Let S be a non empty non void many sorted signature and let f, g be functions. We say that f and g form a replacement in S if and only if the condition (Def. 3) is satisfied.

(Def. 3)   Let o_1, o_2 be operation symbols of S.   Suppose $(\mathrm{id}_{\text{the operation symbols of } S} + \cdot g)(o_1) = (\mathrm{id}_{\text{the operation symbols of } S} + \cdot g)(o_2)$. Then

(i) $(\mathrm{id}_{\text{the carrier of } S} + \cdot f) \cdot \mathrm{Arity}(o_1) = (\mathrm{id}_{\text{the carrier of } S} + \cdot f) \cdot \mathrm{Arity}(o_2)$, and

(ii) $(\mathrm{id}_{\text{the carrier of } S} + \cdot f)(\text{the result sort of } o_1) = (\mathrm{id}_{\text{the carrier of } S} + \cdot f)(\text{the result sort of } o_2)$.

Next we state two propositions:

(30)   Let S be a non empty non void many sorted signature and f, g be functions. Then f and g form a replacement in S if and only if for all operation symbols o_1, o_2 of S such that ((the operation symbols of S)-indexing g)(o_1) = ((the operation symbols of S)-indexing g)(o_2) holds ((the carrier of S)-indexing f) $\cdot$ Arity(o_1) = ((the carrier of S)-indexing f) $\cdot$ Arity(o_2) and ((the carrier of S)-indexing f)(the result sort of o_1) = ((the carrier of S)-indexing f)(the result sort of o_2).

(31)   Let S be a non empty non void many sorted signature and f, g be functions. Then f and g form a replacement in S if and only if (the carrier of S)-indexing f and (the operation symbols of S)-indexing g form a replacement in S.

In the sequel S, S' are non void signatures and f, g are functions.
One can prove the following four propositions:

(32)   If f and g form morphism between S and S', then f and g form a replacement in S.

(33) f and $\emptyset$ form a replacement in S.

(34)   If g is one-to-one and (the operation symbols of S) $\cap$ rng $g \subseteq$ dom g, then f and g form a replacement in S.

(35)   If g is one-to-one and rng g misses the operation symbols of S, then f and g form a replacement in S.

Let X be a set, let Y be a non empty set, let a be a function from Y into X^*, and let r be a function from Y into X. Note that $\langle X, Y, a, r \rangle$ is non void.
Let S be a non empty non void many sorted signature and let f, g be functions. Let us assume that f and g form a replacement in S. The functor S with-replacement(f, g) yields a strict non empty non void many sorted signature and is defined by the conditions (Def. 4).

(Def. 4)(i)   (The carrier of S)-indexing f and (the operation symbols of S)-indexing g form morphism between S and S with-replacement(f, g),

(ii)   the carrier of S with-replacement(f, g) = rng((the carrier of S)-indexing f), and

(iii)   the operation symbols of S with-replacement(f, g) = rng((the operation symbols of S)-indexing g).

We now state a number of propositions:

(36)   Let S_1, S_2 be non void signatures, f be a function from the carrier of S_1 into the carrier of S_2, and g be a function. Suppose f and g form morphism between S_1 and S_2. Then $f^* \cdot$ the arity of S_1 = (the arity of S_2) $\cdot g$.

(37)   Suppose f and g form a replacement in S. Then (the carrier of S)-indexing f is a function from the carrier of S into the carrier of S with-replacement(f, g).

(38)   Suppose f and g form a replacement in S. Let f' be a function from the carrier of S into the carrier of S with-replacement(f, g). Suppose $f' = $ (the carrier of S)-indexing f. Let g' be a rng-retraction of (the operation symbols of S)-indexing g. Then the arity of S with-replacement(f, g) = $f'^* \cdot$ the arity of $S \cdot g'$.

(39)   Suppose f and g form a replacement in S. Let g' be a rng-retraction of (the operation symbols of S)-indexing g. Then the result sort of S with-replacement(f, g) = ((the carrier of S)-indexing f) $\cdot$ the result sort of $S \cdot g'$.

(40)   If f and g form morphism between S and S', then S with-replacement(f, g) is a subsignature of S'.

(41) f and g form a replacement in S if and only if (the carrier of S)-indexing f and (the operation symbols of S)-indexing g form morphism between S and S with-replacement(f, g).

(42) Suppose dom $f \subseteq$ the carrier of S and dom $g \subseteq$ the operation symbols of S and f and g form a replacement in S. Then $\mathrm{id}_{\text{the carrier of } S} \dotplus f$ and $\mathrm{id}_{\text{the operation symbols of } S} \dotplus g$ form morphism between S and S with-replacement(f, g).

(43) Suppose dom $f =$ the carrier of S and dom $g =$ the operation symbols of S and f and g form a replacement in S. Then f and g form morphism between S and S with-replacement(f, g).

(44) If f and g form a replacement in S, then S with-replacement$(($the carrier of $S)$-indexing $f, g) = S$ with-replacement(f, g).

(45) If f and g form a replacement in S, then S with-replacement$(f, ($the operation symbols of $S)$-indexing $g) = S$ with-replacement(f, g).

3. SIGNATURE EXTENSIONS

Let S be a signature. A signature is called an extension of S if:

(Def. 5) S is a subsignature of it.

The following propositions are true:

(47)[1] Every signature S is an extension of S.

(48) For every signature S_1 and for every extension S_2 of S_1 holds every extension of S_2 is an extension of S_1.

(49) For all non empty signatures S_1, S_2 such that $S_1 \approx S_2$ holds $S_1 \dotplus S_2$ is an extension of S_1.

(50) For all non empty signatures S_1, S_2 holds $S_1 \dotplus S_2$ is an extension of S_2.

(51) Let S_1, S_2, S be non empty many sorted signatures and f_1, g_1, f_2, g_2 be functions. Suppose $f_1 \approx f_2$ and f_1 and g_1 form morphism between S_1 and S and f_2 and g_2 form morphism between S_2 and S. Then $f_1 \dotplus f_2$ and $g_1 \dotplus g_2$ form morphism between $S_1 \dotplus S_2$ and S.

(52) Let S_1, S_2, E be non empty signatures. Then E is an extension of S_1 and an extension of S_2 if and only if $S_1 \approx S_2$ and E is an extension of $S_1 \dotplus S_2$.

Let S be a non empty signature. One can verify that every extension of S is non empty.
Let S be a non void signature. Observe that every extension of S is non void.
Next we state the proposition

(53) For all signatures S, T such that S is empty holds T is an extension of S.

Let S be a signature. Note that there exists an extension of S which is non empty, non void, and strict.
We now state three propositions:

(54) Let S be a non void signature and E be an extension of S. Suppose f and g form a replacement in E. Then f and g form a replacement in S.

(55) Let S be a non void signature and E be an extension of S. Suppose f and g form a replacement in E. Then E with-replacement(f, g) is an extension of S with-replacement(f, g).

(56) Let S_1, S_2 be non void signatures. Suppose $S_1 \approx S_2$. Let f, g be functions. If f and g form a replacement in $S_1 \dotplus S_2$, then $(S_1 \dotplus S_2)$ with-replacement$(f, g) = (S_1$ with-replacement$(f, g)) \dotplus (S_2$ with-replacement$(f, g))$.

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[1] The proposition (46) has been removed.

4. ALGEBRAS

Algebra is defined by:

(Def. 6)   There exists a non void signature S such that it is a feasible algebra over S.

Let S be a signature. An algebra is called an algebra of S if:

(Def. 7)   There exists a non void extension E of S such that it is a feasible algebra over E.

Next we state a number of propositions:

(57)   For every non void signature S holds every feasible algebra over S is an algebra of S.

(58)   For every signature S and for every extension E of S holds every algebra of E is an algebra of S.

(59)   Let S be a signature, E be a non empty signature, and A be an algebra over E. Suppose A is an algebra of S. Then the carrier of $S \subseteq$ the carrier of E and the operation symbols of $S \subseteq$ the operation symbols of E.

(60)   Let S be a non void signature, E be a non empty signature, and A be an algebra over E. Suppose A is an algebra of S. Let o be an operation symbol of S. Then (the characteristics of $A)(o)$ is a function from (the sorts of $A)^{\#}(\mathrm{Arity}(o))$ into (the sorts of A)(the result sort of o).

(61)   Let S be a non empty signature, A be an algebra of S, and E be a non empty many sorted signature. If A is an algebra over E, then A is an algebra over $E + \cdot S$.

(62)   Let S_1, S_2 be non empty signatures and A be an algebra over S_1. Suppose A is an algebra over S_2. Then the carrier of $S_1 =$ the carrier of S_2 and the operation symbols of $S_1 =$ the operation symbols of S_2.

(63)   Let S be a non void signature and A be a non-empty disjoint valued algebra over S. Then the sorts of A are one-to-one.

(64)   Let S be a non void signature, A be a disjoint valued algebra over S, and C_1, C_2 be components of the sorts of A. Then $C_1 = C_2$ or C_1 misses C_2.

(65)   Let S, S' be non void signatures and A be a non-empty disjoint valued algebra over S. Suppose A is an algebra over S'. Then the many sorted signature of $S =$ the many sorted signature of S'.

(66)   Let S' be a non void signature and A be a non-empty disjoint valued algebra over S. If A is an algebra of S', then S is an extension of S'.

REFERENCES

[1] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/card_3.html.

[2] Grzegorz Bancerek. Minimal signature for partial algebra. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/pua2mss1.html.

[3] Grzegorz Bancerek. Institution of many sorted algebras. Part I: Signature reduct of an algebra. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/instalg1.html.

[4] Grzegorz Bancerek. Translations, endomorphisms, and stable equational theories. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/msualg_6.html.

[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[6] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[7] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[8] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[9] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[10] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/funct_4.html.

[11] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar — part I. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/lang1.html.

[12] Yatsuka Nakamura and Grzegorz Bancerek. Combining of circuits. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/circcomb.html.

[13] Andrzej Nędzusiak. Probability. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/prob_2.html.

[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[15] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[16] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_1.html.

[17] Andrzej Trybulec. A scheme for extensions of homomorphisms of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msafree1.html.

[18] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[19] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[20] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.