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Homomorphisms of Algebras. Quotient Universal Algebra

Małgorzata Korolkiewicz
Warsaw University
Białystok

Summary. The first part introduces homomorphisms of universal algebras and their basic properties. The second is concerned with the construction of a quotient universal algebra. The first isomorphism theorem is proved.

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The articles [10], [11], [12], [14], [13], [3], [1], [5], [9], [7], [4], [8], [6], and [2] provide the notation and terminology for this paper.

1. HOMOMORPHISMS OF ALGEBRAS

For simplicity, we adopt the following convention: U_1, U_2, U_3 denote universal algebras, n denotes a natural number, o_1 denotes an operation of U_1, o_2 denotes an operation of U_2, and x, y denote sets.

The following two propositions are true:

(1) Let D_1, D_2 be non empty sets, p be a finite sequence of elements of D_1, and f be a function from D_1 into D_2. Then $\mathrm{dom}(f \cdot p) = \mathrm{dom}\, p$ and $\mathrm{len}(f \cdot p) = \mathrm{len}\, p$ and for every n such that $n \in \mathrm{dom}(f \cdot p)$ holds $(f \cdot p)(n) = f(p(n))$.

(2) For every non empty subset B of U_1 such that $B =$ the carrier of U_1 holds $\mathrm{Opers}(U_1, B) =$ the characteristic of U_1.

Let U_1, U_2 be 1-sorted structures. A function from U_1 into U_2 is a function from the carrier of U_1 into the carrier of U_2.

In the sequel a is a finite sequence of elements of U_1 and f is a function from U_1 into U_2.

Next we state three propositions:

(3) $f \cdot \varepsilon_{(\text{the carrier of } U_1)} = \varepsilon_{(\text{the carrier of } U_2)}$.

(4) $\mathrm{id}_{\text{the carrier of } U_1} \cdot a = a$.

(5) Let h_1 be a function from U_1 into U_2, h_2 be a function from U_2 into U_3, and a be a finite sequence of elements of U_1. Then $h_2 \cdot (h_1 \cdot a) = (h_2 \cdot h_1) \cdot a$.

Let us consider U_1, U_2, f. We say that f is a homomorphism of U_1 into U_2 if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) U_1 and U_2 are similar, and

(ii) for every n such that $n \in \mathrm{dom}$(the characteristic of U_1) and for all o_1, o_2 such that $o_1 =$ (the characteristic of U_1)(n) and $o_2 =$ (the characteristic of U_2)(n) and for every finite sequence x of elements of U_1 such that $x \in \mathrm{dom}\, o_1$ holds $f(o_1(x)) = o_2(f \cdot x)$.

Let us consider U_1, U_2, f. We say that f is a monomorphism of U_1 into U_2 if and only if:

(Def. 2) f is a homomorphism of U_1 into U_2 and one-to-one.

We say that f is an epimorphism of U_1 onto U_2 if and only if:

(Def. 3) f is a homomorphism of U_1 into U_2 and $\mathrm{rng}\, f =$ the carrier of U_2.

Let us consider U_1, U_2, f. We say that f is an isomorphism of U_1 and U_2 if and only if:

(Def. 4) f is a monomorphism of U_1 into U_2 and an epimorphism of U_1 onto U_2.

Let us consider U_1, U_2. We say that U_1 and U_2 are isomorphic if and only if:

(Def. 5) There exists f which is an isomorphism of U_1 and U_2.

The following propositions are true:

(6) $\mathrm{id}_{\text{the carrier of } U_1}$ is a homomorphism of U_1 into U_1.

(7) Let h_1 be a function from U_1 into U_2 and h_2 be a function from U_2 into U_3. Suppose h_1 is a homomorphism of U_1 into U_2 and h_2 is a homomorphism of U_2 into U_3. Then $h_2 \cdot h_1$ is a homomorphism of U_1 into U_3.

(8) f is an isomorphism of U_1 and U_2 if and only if f is a homomorphism of U_1 into U_2 and $\mathrm{rng}\, f =$ the carrier of U_2 and f is one-to-one.

(9) If f is an isomorphism of U_1 and U_2, then $\mathrm{dom}\, f =$ the carrier of U_1 and $\mathrm{rng}\, f =$ the carrier of U_2.

(10) Let h be a function from U_1 into U_2 and h_1 be a function from U_2 into U_1. Suppose h is an isomorphism of U_1 and U_2 and $h_1 = h^{-1}$. Then h_1 is a homomorphism of U_2 into U_1.

(11) Let h be a function from U_1 into U_2 and h_1 be a function from U_2 into U_1. Suppose h is an isomorphism of U_1 and U_2 and $h_1 = h^{-1}$. Then h_1 is an isomorphism of U_2 and U_1.

(12) Let h be a function from U_1 into U_2 and h_1 be a function from U_2 into U_3. Suppose h is an isomorphism of U_1 and U_2 and h_1 is an isomorphism of U_2 and U_3. Then $h_1 \cdot h$ is an isomorphism of U_1 and U_3.

(13) U_1 and U_1 are isomorphic.

(14) If U_1 and U_2 are isomorphic, then U_2 and U_1 are isomorphic.

(15) If U_1 and U_2 are isomorphic and U_2 and U_3 are isomorphic, then U_1 and U_3 are isomorphic.

Let us consider U_1, U_2, f. Let us assume that f is a homomorphism of U_1 into U_2. The functor $\mathrm{Im}\, f$ yielding a strict subalgebra of U_2 is defined by:

(Def. 6) The carrier of $\mathrm{Im}\, f = f^{\circ}$(the carrier of U_1).

One can prove the following two propositions:

(16) For every function h from U_1 into U_2 such that h is a homomorphism of U_1 into U_2 holds $\mathrm{rng}\, h =$ the carrier of $\mathrm{Im}\, h$.

(17) Let U_2 be a strict universal algebra and f be a function from U_1 into U_2. Suppose f is a homomorphism of U_1 into U_2. Then f is an epimorphism of U_1 onto U_2 if and only if $\mathrm{Im}\, f = U_2$.

2. QUOTIENT UNIVERSAL ALGEBRA

Let U_1 be a 1-sorted structure. A binary relation on U_1 is a binary relation on the carrier of U_1. An equivalence relation of U_1 is an equivalence relation of the carrier of U_1.

Let D be a non empty set and let R be a binary relation on D. The functor $R^\#$ yields a binary relation on D^* and is defined by the condition (Def. 9).

(Def. 9)[1] Let x, y be finite sequences of elements of D. Then $\langle x, y\rangle \in R^\#$ if and only if the following conditions are satisfied:

(i) $\operatorname{len} x = \operatorname{len} y$, and

(ii) for every n such that $n \in \operatorname{dom} x$ holds $\langle x(n), y(n)\rangle \in R$.

We now state the proposition

(18) For every non empty set D holds $(\mathrm{id}_D)^\# = \mathrm{id}_{D^*}$.

Let us consider U_1. An equivalence relation of U_1 is said to be a congruence of U_1 if it satisfies the condition (Def. 10).

(Def. 10) Let given n, o_1. Suppose $n \in \operatorname{dom}$(the characteristic of U_1) and $o_1 =$ (the characteristic of $U_1)(n)$. Let x, y be finite sequences of elements of U_1. If $x \in \operatorname{dom} o_1$ and $y \in \operatorname{dom} o_1$ and $\langle x, y\rangle \in \mathrm{it}^\#$, then $\langle o_1(x), o_1(y)\rangle \in$ it.

In the sequel E denotes a congruence of U_1.

Let D be a non empty set, let R be an equivalence relation of D, let y be a finite sequence of elements of Classes R, and let x be a finite sequence of elements of D. We say that x is a finite sequence of representatives of y if and only if:

(Def. 11) $\operatorname{len} x = \operatorname{len} y$ and for every n such that $n \in \operatorname{dom} x$ holds $[x(n)]_R = y(n)$.

The following proposition is true

(19) Let D be a non empty set, R be an equivalence relation of D, and y be a finite sequence of elements of Classes R. Then there exists a finite sequence of elements of D which is a finite sequence of representatives of y.

Let U_1 be a universal algebra, let E be a congruence of U_1, and let o be an operation of U_1. The functor $o_{/E}$ yields a homogeneous quasi total non empty partial function from $(\mathrm{Classes}\,E)^*$ to Classes E and is defined by the conditions (Def. 12).

(Def. 12)(i) $\operatorname{dom}(o_{/E}) = (\mathrm{Classes}\,E)^{\operatorname{arity} o}$, and

(ii) for every finite sequence y of elements of Classes E such that $y \in \operatorname{dom}(o_{/E})$ and for every finite sequence x of elements of the carrier of U_1 such that x is a finite sequence of representatives of y holds $o_{/E}(y) = [o(x)]_E$.

Let us consider U_1, E. The functor $\mathrm{Opers}((U_1))_{/E}$ yields a finite sequence of operational functions of Classes E and is defined by the conditions (Def. 13).

(Def. 13)(i) $\operatorname{len}(\mathrm{Opers}((U_1))_{/E}) = \operatorname{len}$(the characteristic of U_1), and

(ii) for every n such that $n \in \operatorname{dom}(\mathrm{Opers}((U_1))_{/E})$ and for every o_1 such that (the characteristic of $U_1)(n) = o_1$ holds $\mathrm{Opers}((U_1))_{/E}(n) = (o_1)_{/E}$.

We now state the proposition

(20) For all U_1, E holds $\langle \mathrm{Classes}\,E, \mathrm{Opers}((U_1))_{/E}\rangle$ is a strict universal algebra.

Let us consider U_1, E. The functor $(U_1)_{/E}$ yielding a strict universal algebra is defined as follows:

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[1] The definitions (Def. 7) and (Def. 8) have been removed.

(Def. 14) $(U_1)_{/E} = \langle \mathrm{Classes}\, E, \mathrm{Opers}((U_1))_{/E} \rangle$.

Let us consider U_1, E. The natural homomorphism of U_1 w.r.t. E yields a function from U_1 into $(U_1)_{/E}$ and is defined as follows:

(Def. 15) For every element u of U_1 holds (the natural homomorphism of U_1 w.r.t. $E)(u) = [u]_E$.

We now state two propositions:

(21) For all U_1, E holds the natural homomorphism of U_1 w.r.t. E is a homomorphism of U_1 into $(U_1)_{/E}$.

(22) For all U_1, E holds the natural homomorphism of U_1 w.r.t. E is an epimorphism of U_1 onto $(U_1)_{/E}$.

Let us consider U_1, U_2 and let f be a function from U_1 into U_2. Let us assume that f is a homomorphism of U_1 into U_2. The functor $\mathrm{Cng}(f)$ yielding a congruence of U_1 is defined by:

(Def. 16) For all elements a, b of U_1 holds $\langle a, b \rangle \in \mathrm{Cng}(f)$ iff $f(a) = f(b)$.

Let U_1, U_2 be universal algebras and let f be a function from U_1 into U_2. Let us assume that f is a homomorphism of U_1 into U_2. The functor $\overline{f}$ yielding a function from $(U_1)_{/\mathrm{Cng}(f)}$ into U_2 is defined by:

(Def. 17) For every element a of U_1 holds $(\overline{f})([a]_{\mathrm{Cng}(f)}) = f(a)$.

Next we state three propositions:

(23) Suppose f is a homomorphism of U_1 into U_2. Then $\overline{f}$ is a homomorphism of $(U_1)_{/\mathrm{Cng}(f)}$ into U_2 and $\overline{f}$ is a monomorphism of $(U_1)_{/\mathrm{Cng}(f)}$ into U_2.

(24) If f is an epimorphism of U_1 onto U_2, then $\overline{f}$ is an isomorphism of $(U_1)_{/\mathrm{Cng}(f)}$ and U_2.

(25) If f is an epimorphism of U_1 onto U_2, then $(U_1)_{/\mathrm{Cng}(f)}$ and U_2 are isomorphic.

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